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PLANE TRIGONOMETRY

BY

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SOMETIME FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

PART 1.

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PREFACE.

THE following work will, I hope, be found to be a fairly complete elementary text-book on Plane Trigonometry, suitable for Schools and the Pass and Junior Honour classes of Universities. In the higher portion of the book I have endeavoured to present to the student, as simply as possible, the modern treatment of complex quantities, and I hope it will be found that he will have little to unlearn when he commences to read treatises of a more difficult character.

As Trigonometry consists largely of formulæ and the applications thereof, I have prefixed a list of the principal formulæ which the student should commit to memory. These more important formulæ are distinguished in the text by the use of thick type. Other formulæ are subsidiary and of less importance.

The number of examples is very large. A selection only should be solved by the student on a first reading.
On a first reading also the articles marked with an asterisk should be omitted.

Considerable attention has been paid to the printing of the book and I am under great obligation to the Syndics of the Press for their liberality in this matter, and to the officers and workmen of the Press for the trouble they have taken.

I am indebted to Mr W. J. Dobbs, B.A., late Scholar of St John's College, for his kindness in reading and correcting the proof-sheets and for many valuable suggestions.

For any corrections and suggestions for improvement I shall be thankful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE,
EGHAM, SURREY.
September 12, 1893.

PREFACE TO THE SECOND EDITION.

The Second Edition has been carefully revised, and it is hoped that few serious mistakes remain either in the text or the answers.

Some changes have been made in the chapters on logarithms and logarithmic tables, and an additional chapter has been added on Projections.

April 25, 1895.
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THE PRINCIPAL FORMULÆ IN TRIGONOMETRY.

PART I.

I. Circumference of a circle = $2\pi r$. (Art. 12.)

\[ \pi = 3.14159 \ldots \left[ \text{Approximations are } \frac{22}{7} \text{ and } \frac{355}{113} \right]. \] (Art. 13.)

A Radian = $57^\circ 17' 44.8''$ nearly. (Art. 16.)

Two right angles = $180^\circ = 200^g = \pi$ radians. (Art. 19.)

Angle = $\frac{\text{arc}}{\text{radius}} \times \text{Radian}$. (Art. 21.)

II. \[ \sin^2 \theta + \cos^2 \theta = 1; \]

\[ \sec^2 \theta = 1 + \tan^2 \theta; \]

\[ \cosec^2 \theta = 1 + \cot^2 \theta. \] (Art. 27.)

III. \[ \sin 0^\circ = 0; \ \cos 0^\circ = 1. \] (Art. 36.)

\[ \sin 30^\circ = \frac{1}{2}; \ \cos 30^\circ = \frac{\sqrt{3}}{2}. \] (Art. 34.)

\[ \sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}. \] (Art. 33.)

\[ \sin 60^\circ = \frac{\sqrt{3}}{2}; \ \cos 60^\circ = \frac{1}{2}. \] (Art. 35.)

\[ \sin 90^\circ = 1; \ \cos 90^\circ = 0. \] (Art. 37.)

\[ \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \ \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}. \] (Art. 106.)

\[ \sin 18^\circ = \frac{\sqrt{5} - 1}{4}; \ \cos 36^\circ = \frac{\sqrt{5} + 1}{4}. \] (Arts. 120, 121.)
THE PRINCIPAL FORMULÆ IN TRIGONOMETRY.

IV. \[ \sin(-\theta) = -\sin\theta; \quad \cos(-\theta) = \cos\theta. \] (Art. 68.)
\[ \sin(90^\circ - \theta) = \cos\theta; \quad \cos(90^\circ - \theta) = \sin\theta. \] (Art. 69.)
\[ \sin(90^\circ + \theta) = \cos\theta; \quad \cos(90^\circ + \theta) = -\sin\theta. \] (Art. 70.)
\[ \sin(180^\circ - \theta) = \sin\theta; \quad \cos(180^\circ - \theta) = -\cos\theta. \] (Art. 72.)
\[ \sin(180^\circ + \theta) = -\sin\theta; \quad \cos(180^\circ + \theta) = -\cos\theta. \] (Art. 73.)

V. If \( \sin \theta = \sin a \), then \( \theta = n\pi + (-1)^n a \). (Art. 82.)
If \( \cos \theta = \cos a \), then \( \theta = 2n\pi \pm a. \) (Art. 83.)
If \( \tan \theta = \tan a \), then \( \theta = n\pi + a. \) (Art. 84.)

VI. \[ \sin (A + B) = \sin A \cos B + \cos A \sin B. \] (Art. 88.)
\[ \cos (A + B) = \cos A \cos B - \sin A \sin B. \] (Art. 88.)
\[ \sin (A - B) = \sin A \cos B - \cos A \sin B. \]
\[ \cos (A - B) = \cos A \cos B + \sin A \sin B. \] (Art. 90.)
\[ \sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}. \]
\[ \sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}. \]
\[ \cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}. \]
\[ \cos D - \cos C = 2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}. \] (Art. 94.)
\[ 2 \sin A \cos B = \sin (A + B) + \sin (A - B) \]
\[ 2 \cos A \sin B = \sin (A + B) - \sin (A - B). \]
\[ 2 \cos A \cos B = \cos (A + B) + \cos (A - B). \]
\[ 2 \sin A \sin B = \cos (A - B) - \cos (A + B). \] (Art. 97.)
\[\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.\]

\[\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.\] (Art. 98.)

\[\sin 2A = 2 \sin A \cos A.\]
\[\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1.\] (Art. 105).

\[\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}; \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}.\] (Art. 109.)

\[\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.\] (Art. 105.)

\[\sin 3A = 3 \sin A - 4 \sin^3 A.\]
\[\cos 3A = 4 \cos^3 A - 3 \cos A.\]

\[\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.\] (Art. 107.)

\[\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}; \quad \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}.\] (Art. 110.)

\[2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} = \pm \sqrt{1 - \sin A}.\]
\[2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} = \sqrt{1 - \sin A}.\] (Art. 113.)

\[\tan (A_1 + A_2 + \ldots + A_n) = \frac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots}.\] (Art. 125.)

**VII.**

\[\log_a mn = \log_a m + \log_a n.\]

\[\log_a \frac{m}{n} = \log_a m - \log_a n.\]

\[\log_a m^n = n \log_a m.\] (Art. 136.)

\[\log_a m = \log_b m \times \log_a b.\] (Art. 147.)
THE PRINCIPAL FORMULÆ IN TRIGONOMETRY.

\[ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \quad \text{(Art. 163.)} \]
\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ldots \quad \text{(Art. 164.)} \]
\[ \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \ldots \quad \text{(Art. 165.)} \]
\[ \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \ldots \quad \text{(Art. 166.)} \]
\[ \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \ldots \quad \text{(Art. 167.)} \]
\[ \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}, \ldots \quad \text{(Art. 169.)} \]
\[ a = b \cos C + c \cos B, \ldots \quad \text{(Art. 170.)} \]
\[ \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}, \ldots \quad \text{(Art. 171.)} \]

\[ S = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C. \quad \text{(Art. 198.)} \]

\[ R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4S}. \quad \text{(Arts. 200, 201.)} \]
\[ r = \frac{S}{s} = (s-a) \tan \frac{A}{2} = \ldots \quad \text{(Arts. 202, 203.)} \]
\[ r_1 = \frac{S}{s-a} = s \tan \frac{A}{2}. \quad \text{(Arts. 205, 206.)} \]

**Area of a quadrilateral inscribable in a circle**
\[ \sqrt{(s-a)(s-b)(s-c)(s-d)}. \quad \text{(Art. 219.)} \]
\[ \frac{\sin \theta}{\theta} = 1, \text{ when } \theta \text{ is very small.} \quad \text{(Art. 228.)} \]

**Area of a circle**
\[ = \pi r^2. \quad \text{(Art. 233.)} \]
$X.$ \hspace{1cm} \sin a + \sin (a + \beta) + \sin (a + 2\beta) + \ldots \text{ to } n \text{ terms}

\sin \left\{a + \frac{n - 1}{2} \beta\right\} \sin \frac{n\beta}{2} = \frac{\sin \frac{\beta}{2}}{\sin \frac{n\beta}{2}}. \quad (\text{Art. 241.})

\cos a + \cos (a + \beta) + \cos (a + 2\beta) + \ldots \text{ to } n \text{ terms}

\cos \left\{a + \frac{n - 1}{2} \beta\right\} \sin \frac{n\beta}{2} = \frac{\sin \frac{\beta}{2}}{\sin \frac{n\beta}{2}}. \quad (\text{Art. 242.})
CHAPTER I.

MEASUREMENT OF ANGLES, SEXAGESIMAL, CENTESIMAL, AND CIRCULAR MEASURE.

1. In geometry angles are measured in terms of a right angle. This, however, is an inconvenient unit of measurement on account of its size.

2. In the Sexagesimal system of measurement a right angle is divided into 90 equal parts called Degrees. Each degree is divided into 60 equal parts called Minutes, and each minute into 60 equal parts called Seconds.

The symbols 1°, 1′, and 1″ are used to denote a degree, a minute, and a second respectively.

Thus 60 Seconds (60″) make One Minute (1′),
   60 Minutes (60′) " " " Degree (1°),
and 90 Degrees (90°) " " " Right Angle.

This system is well established and is always used in the practical applications of Trigonometry. It is not however very convenient on account of the multipliers 60 and 90.

L. T.
3. On this account another system of measurement called the Centesimal, or French, system has been proposed. In this system the right angle is divided into 100 equal parts, called Grades; each grade is subdivided into 100 Minutes, and each minute into 100 Seconds.

The symbols 1°, 1', and 1'' are used to denote a Grade, a Minute, and a Second respectively.

Thus 100 Seconds (100") make One Minute (1'),
100 Minutes (100') " " Grade, (1°),
100 Grades (100°) " " Right angle.

4. This system would be much more convenient to use than the ordinary Sexagesimal System.

As a preliminary, however, to its practical adoption, a large number of tables would have to be recalculated. For this reason the system has in practice never been used.

5. To convert Sexagesimal into Centesimal Measure, and vice versa.

Since a right angle is equal to 90° and also to 100°, we have

\[ 90° = 100°. \]
\[ \therefore 1° = \frac{10°}{9}, \text{ and } 1° = \frac{9°}{10}. \]

Hence, to change degrees into grades, add on one-ninth; to change grades into degrees, subtract one-tenth.

**Ex.**

\[ 36° = \left(36 + \frac{1}{9} \times 36\right)° = 40°, \]
and
\[ 64° = \left(64 - \frac{1}{10} \times 64\right)° = (64 - 6.4)° = 57.6°. \]

If the angle do not contain an integral number of degrees, we may reduce it to a fraction of a degree and then change to grades.
In practice it is generally found more convenient to reduce any angle to a fraction of a right angle. The method will be seen in the following examples;

**Ex. 1.** Reduce $63^\circ 14' 51''$ to Centesimal Measure.

We have 
\[ 51'' = \frac{17'}{20} = '85', \]
and 
\[ 14' 51'' = 14 \cdot 85' = \frac{14 \cdot 85^\circ}{60} = ^\circ '2475', \]
\[ \therefore 63^\circ 14' 51'' = 63 \cdot 2475^\circ = \frac{63 \cdot 2475}{90} \text{ rt. angle} \]
\[ = 70275 \text{ rt. angle} \]
\[ = 70 \cdot 275^\circ = 70^\circ 27' 50'' = 70^\circ 27' 50''. \]

**Ex. 2.** Reduce $94^\circ 23' 87''$ to Sexagesimal Measure.

$94^\circ 23' 87'' = 94^\circ 23' 87''$ right angle

\[ \begin{align*}
90 & \quad \begin{array}{c}
\text{degrees}
\end{array} \\
84.81483 & \quad \begin{array}{c}
\text{minutes}
\end{array} \\
48.8993 & \quad \begin{array}{c}
\text{seconds}
\end{array} \\
60 & \\
53.3880 & \\
\end{align*} \]
\[ \therefore 94^\circ 23' 87'' = 84^\circ 48' 53.388''. \]

6. **Angles of any size.**

Suppose $AOA'$ and $BOB'$ to be two fixed lines meeting at right angles in $O$, and suppose a revolving line $OP$ (turning about a fixed point at $O$) to start from $OA$ and revolve in a direction opposite to that of the hands of a watch.

For any position of the revolving line between $OA$ and $OB$, such as $OP_1$, it will have turned through an angle $AOP_1$, which is less than a right angle.
TRIGONOMETRY.

For any position between \( OB \) and \( OA' \), such as \( OP_2 \), the angle \( AOP_2 \) through which it has turned is greater than a right angle.

For any position \( OP_3 \), between \( OA' \) and \( OB' \), the angle traced out is \( AOP_3 \), i.e. \( AOB + BOA' + A'OP_3 \), i.e. 2 right angles + \( A'OP_3 \), so that the angle described is greater than two right angles.

For any position \( OP_4 \), between \( OB' \) and \( OA \), the angle turned through is similarly greater than three right angles.

When the revolving line has made a complete revolution, so that it coincides once more with \( OA \), the angle through which it has turned is 4 right angles.

If the line \( OP \) still continue to revolve, the angle through which it has turned, when it is for the second time in the position \( OP_1 \), is not \( AOP_1 \) but 4 right angles + \( AOP_1 \).

Similarly, when the revolving line, having made two complete revolutions, is once more in the position \( OP_2 \), the angle it has traced out is 8 right angles + \( AOP_2 \).

7. If the revolving line \( OP \) be between \( OA \) and \( OB \), it is said to be in the first quadrant; if it be between \( OB \) and \( OA' \), it is in the second quadrant; if between \( OA' \) and \( OB' \), it is in the third quadrant; if it is between \( OB' \) and \( OA \), it is in the fourth quadrant.

8. Ex. What is the position of the revolving line when it has turned through (1) \( 225^\circ \), (2) \( 480^\circ \), and (3) \( 1050^\circ \)?

(1) Since \( 225^\circ = 180^\circ + 45^\circ \), the revolving line has turned through 45° more than two right angles, and it is therefore in the third quadrant and halfway between \( OA' \) and \( OB' \).

(2) Since \( 480^\circ = 360^\circ + 120^\circ \), the revolving line has turned through 120° more than one complete revolution, and is therefore in the second quadrant, i.e. between \( OB \) and \( OA' \), and makes an angle of 30° with \( OB \).
(3) Since $1050^\circ = 11 \times 90^\circ + 60^\circ$, the revolving line has turned through $60^\circ$ more than eleven right angles, and is therefore in the fourth quadrant, i.e. between $OB'$ and $OA$, and makes $60^\circ$ with $OB'$.

**EXAMPLES. I.**

Express in terms of a right angle the angles

1. $60^\circ$.  
2. $75^\circ 15'$.  
3. $63^\circ 17' 25''$.  
4. $130^\circ 30'$.  
5. $210^\circ 30' 30''$.  
6. $370^\circ 20' 48''$.

Express in grades, minutes, and seconds the angles

7. $30^\circ$.  
8. $81^\circ$.  
9. $138^\circ 30'$.  
10. $35^\circ 47' 16''$.  
11. $235^\circ 12' 36''$.  
12. $475^\circ 13' 48''$.

Express in terms of right angles, and also in degrees, minutes, and seconds the angles

13. $120^\circ$.  
14. $45^\circ 35' 24''$.  
15. $39^\circ 45' 36''$.  
16. $255^\circ 8' 9''$.  
17. $753^\circ 0' 5''$.

Mark the position of the revolving line when it has traced out the following angles:

18. $\frac{4}{3}$ right angle.  
19. $3\frac{1}{2}$ right angles.  
20. $13\frac{1}{2}$ right angles.

21. $120^\circ$.  
22. $315^\circ$.  
23. $745^\circ$.  
24. $1185^\circ$.  
25. $150^\circ$.  
26. $420^\circ$.  
27. $875^\circ$.

28. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{4}$ minutes by the hour and minute hands of a watch?

29. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both the angles in degrees.

30. Prove that the number of Sexagesimal minutes in any angle is to the number of Centesimal minutes in the same angle as 27 : 50.

31. Divide $44^\circ 8'$ into two parts such that the number of Sexagesimal seconds in one part may be equal to the number of Centesimal seconds in the other part.

**Circular Measure.**

9. A third system of measurement of angles has been devised, and it is this system which is used in all the higher branches of Mathematics.
The unit used is obtained thus;
Take any circle \( A\!P\!B\!B' \), whose centre is \( O \), and from any point \( A \) measure off an arc \( A\!P \) whose length is equal to the radius of the circle. Join \( OA \) and \( OP \).

The angle \( A\!O\!P \) is the angle which is taken as the unit of circular measurement, \( i.e. \) it is the angle in terms of which in this system we measure all others.

This angle is called a **Radian** and is often denoted by \( 1^\circ \).

10. It is clearly essential to the proper choice of a unit that it should be a *constant* quantity; hence we must shew that the Radian is a constant angle. This we shall do in the following articles.

11. **Theorem.** The length of the circumference of a circle always bears a constant ratio to its diameter.

Take any two circles whose common centre is \( O \). In the large circle inscribe a regular polygon of \( n \) sides, \( A\!B\!C\!D \!\ldots \).

Let \( OA, OB, OC, \ldots \) meet the smaller circle in the points \( a, b, c, d \ldots \) and join \( ab, bc, cd, \ldots \).

Then, by Euc. vi. 2, \( abcd \ldots \) is a regular polygon of \( n \) sides inscribed in the smaller circle.

Since \( Oa = Ob \), and \( OA = OB \),
the lines $ab$ and $AB$ must be parallel, and hence
\[
\frac{AB}{ab} = \frac{OA}{Oa} \quad \text{(Eucl. VI. 4)}.
\]

Also the polygon $ABCD\ldots$ being regular, its perimeter, i.e. the sum of its sides, is equal to $n \cdot AB$. Similarly for the inner polygon.

Hence we have
\[
\text{Perimeter of the outer polygon} = \frac{n \cdot AB}{n \cdot ab} = \frac{AB}{ab} = \frac{OA}{Oa}
\]

\[\text{..............}(1)\]

This relation exists whatever be the number of sides in the polygons.

Let then the number of sides be indefinitely increased (i.e. let $n$ become inconceivably great) so that finally the perimeter of the outer polygon will be the same as the circumference of the outer circle, and the perimeter of the inner polygon the same as the circumference of the inner circle.

The relation (1) will then become
\[
\frac{\text{Circumference of outer circle}}{\text{Circumference of inner circle}} = \frac{OA}{Oa}
\]

\[= \frac{\text{Radius of outer circle}}{\text{Radius of inner circle}}.
\]

Hence
\[
\frac{\text{Circumference of outer circle}}{\text{Radius of outer circle}} = \frac{\text{Circumference of inner circle}}{\text{Radius of inner circle}}.
\]

Since there was no restriction whatever as to the sizes of the two circles, it follows that the quantity
\[
\frac{\text{Circumference of a circle}}{\text{Radius of the circle}}
\]

is the same for all circles.
Hence the ratio of the circumference of a circle to its radius, and therefore also to its diameter, is a constant quantity.

12. In the previous article we have shewn that the ratio \( \frac{\text{Circumference}}{\text{Diameter}} \) is the same for all circles. The value of this constant ratio is always denoted by the Greek letter \( \pi \) (pronounced Pi), so that \( \pi \) is a number.

Hence \( \frac{\text{Circumference}}{\text{Diameter}} = \text{the constant number } \pi \).

We have therefore the following theorem; The circumference of a circle is always equal to \( \pi \) times its diameter or \( 2\pi \) times its radius.

13. Unfortunately the value of \( \pi \) is not a whole number, nor can it be expressed in the form of a vulgar fraction, and hence not in the form of a decimal fraction, terminating or recurring.

The number \( \pi \) is an incommensurable magnitude, i.e. a magnitude whose value cannot be exactly expressed as the ratio of two whole numbers.

Its value, correct to 8 places of decimals, is

\[ 3.14159265\ldots \]

The fraction \( \frac{22}{7} \) gives the value of \( \pi \) correctly for the first two decimal places; for \( \frac{22}{7} = 3.14285\ldots \).

The fraction \( \frac{355}{113} \) is a more accurate value of \( \pi \), being correct to 6 places of decimals; for \( \frac{355}{113} = 3.14159203\ldots \).
[N.B. The fraction \( \frac{355}{113} \) may be remembered thus; write down the first three odd numbers repeating each twice, thus 113355; divide the number thus obtained into two parts and let the first part be divided into the second, thus 113) 355.

The quotient is the value of \( \pi \) to 6 places of decimals.]

To sum up. **An approximate value of \( \pi \), correct to 2 places of decimals, is the fraction \( \frac{22}{7} \); a more accurate value is 3·14159...**

By division, we can shew that

\[
\frac{1}{\pi} = \cdot3183098862...
\]

**14. Ex. 1.** The diameter of a tricycle wheel is 28 inches; through what distance does its centre move during one revolution of the wheel?

The radius \( r \) is here 14 inches.
The circumference therefore = \( 2 \cdot \pi \cdot 14 = 28\pi \) inches.

If we take \( \pi = \frac{22}{7} \), the circumference \( = 28 \times \frac{22}{7} \) inches = 7 ft. 4 inches approximately.

If we give \( \pi \) the more accurate value 3·14159265..., the circumference

\( = 28 \times 3·14159265... \) inches = 7 ft. 3·96459... inches.

**Ex. 2.** What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe one mile?

The circumference must be \( \frac{1}{5} \times 1760 \), i.e. 352, yards.

Hence, if \( r \) be the radius of the path in yards, we have \( 2\pi r = 352 \), i.e.

\[
r = \frac{176}{\pi} \text{ yards.}
\]

Taking \( \pi = \frac{22}{7} \), we have \( r = \frac{176 \times 7}{22} = 56 \) yards nearly.

Taking the more accurate value \( \frac{1}{\pi} = \cdot31831 \), we have

\[
r = 176 \times \cdot31831 = 56.02256 \text{ yards.}
\]
EXAMPLES. II.

1. If the radius of the earth be 4000 miles, what is the length of its circumference?

2. The wheel of a railway carriage is 3 feet in diameter and makes 3 revolutions in a second; how fast is the train going?

3. A mill sail whose length is 18 feet makes 10 revolutions per minute. What distance does its end travel in an hour?

4. The diameter of a halfpenny is an inch; what is the length of a piece of string which would just surround its curved edge?

5. Assuming that the earth describes in one year a circle, of 92500000 miles radius, whose centre is the sun, how many miles does the earth travel in a year?

6. The radius of a carriage wheel is 1 ft. 9 ins., and in $\frac{1}{9}$ th of a second it turns through 80° about its centre, which is fixed; how many miles does a point on the rim of the wheel travel in one hour?

15. Theorem. The radian is a constant angle.

Take the figure of Art. 9. Let the arc $AB$ be a quadrant of the circle, i.e. one quarter of the circumference.

By Art. 12, the length of $AB$ is therefore $\frac{\pi r}{2}$, where $r$ is the radius of the circle.

By Euc. VI. 33, we know that angles at the centre of any circle are to one another as the arcs on which they stand.

Hence $\frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{r}{\pi} = \frac{2}{\pi}$, $\frac{r}{2}$.

i.e. $\angle AOP = \frac{2}{\pi} \cdot \angle AOB$.

But we defined the angle $AOP$ to be a Radian.
Hence a Radian = \( \frac{2}{\pi} \cdot \angle AOB \)

= \( \frac{2}{\pi} \) of a right angle.

Since a right angle is a constant angle, and since we have shewn (Art. 12) that \( \pi \) is a constant quantity, it follows that a Radian is a constant angle, and is therefore the same whatever be the circle from which it is derived.

\[ \sim \]

16. **Magnitude of a Radian.**

By the previous article, a Radian

\[ \frac{2}{\pi} \times \text{a right angle} = \frac{180^\circ}{\pi} \]

= \( 180^\circ \times 0.3183098862... = 57.2957795^\circ \)

= \( 57^\circ 17' 44.8'' \) nearly.

17. Since a Radian = \( \frac{2}{\pi} \) of a right angle,

therefore a right angle = \( \frac{\pi}{2} \). radians,

so that \( 180^\circ = 2 \) right angles = \( \pi \) radians,

and \( 360^\circ = 4 \) right angles = \( 2\pi \) radians.

Hence, when the revolving line (Art. 6) has made a complete revolution, it has described an angle equal to \( 2\pi \) radians; when it has made three complete revolutions, it has described an angle of \( 6\pi \) radians; when it has made \( n \) revolutions, it has described an angle of \( 2n\pi \) radians.

18. In practice the symbol \( \pi \) is generally omitted, and instead of “an angle \( \pi^\circ \)” we find written “an angle \( \pi \).”
The student must notice this point carefully. If the unit, in terms of which the angle is measured, be not mentioned, he must mentally supply the word "radians." Otherwise he will easily fall into the mistake of supposing that \( \pi \) stands for 180°. It is true that \( \pi \) radians \( (\pi^\circ) \) is the same as 180°, but \( \pi \) itself is a number, and a number only.

19. To convert circular measure into sexagesimal measure or centesimal measure and vice versa.

The student should remember the relations

Two right angles = 180° = 200° = \( \pi \) radians.

The conversion is then merely Arithmetica.

**Ex.**

1. \( 45^\circ \times \pi \approx 45 \times 180^\circ = 81^\circ = 90^\circ \).

2. \( 3^\circ = \frac{3}{\pi} \times \pi^\circ = \frac{3}{\pi} \times 180^\circ = \frac{3}{\pi} \times 200^\circ \).

3. \( 40^\circ 15' 36'' = 40^\circ 15' 200^\circ = 40.26^\circ \)

\[ = 40.26 \times \frac{\pi}{180} = 0.2236\pi \text{ radians.} \]

4. \( 40^\circ 15' 36'' = 40.1536^\circ = 40.1536 \times \frac{\pi}{200} \text{ radians} \)

\[ = 0.200768\pi \text{ radians.} \]

20. **Ex. 1.** The angles of a triangle are in A. P. and the number of grades in the least is to the number of radians in the greatest as 40 : \( \pi \); find the angles in degrees.

Let the angles be \( (x - y)^\circ, x^\circ, \) and \( (x + y)^\circ \).

Since the sum of the three angles of a triangle is 180°, we have

\[ 180 = x - y + x + x + y = 3x, \]

so that \( x = 60. \)

The required angles are therefore

\( (60 - y)^\circ, 60^\circ, \) and \( (60 + y)^\circ. \)

Now

\[ (60 - y)^\circ = \frac{10}{9} \times (60 - y)^\circ, \]

and

\[ (60 + y)^\circ = \frac{\pi}{180} \times (60 + y) \text{ radians.} \]
Hence \[
\frac{10}{9} (60 - y) : \frac{\pi}{180} (60 + y) :: 40 : \pi,
\]
\[\therefore \quad \frac{200}{\pi} \frac{60 - y}{60 + y} = \frac{40}{\pi},\]
i.e. \[5 (60 - y) = 60 + y,\]
i.e. \[y = 40.\]

The angles are therefore 20°, 60°, and 100°.

**Ex. 2.** Express in the 3 systems of angular measurement the magnitude of the angle of a regular decagon.

The corollary to Eucl. I. 32 states that all the interior angles of any rectilinear figure together with four right angles are equal to twice as many right angles as the figure has sides.

Let the angle of a regular decagon contain \(x\) right angles, so that all the angles are together equal to 10\(x\) right angles.

The corollary therefore states that

\[10x + 4 = 20,\]

so that \[x = \frac{8}{5}\] right angles.

But one right angle \[= 90° = 100° = \frac{\pi}{2}\) radians.

Hence the required angle \[= 144° = 160° = \frac{4\pi}{5}\) radians.

**EXAMPLES. III**

Express in degrees, minutes, and seconds the angles,

1. \(\frac{\pi^\circ}{3}\)  2. \(\frac{4\pi^\circ}{3}\)  3. \(10\pi^\circ\)  4. \(1^\circ\)  5. \(8^\circ\).

Express in grades, minutes, and seconds the angles,

6. \(\frac{4\pi^\circ}{5}\)  7. \(\frac{7\pi^\circ}{6}\)  8. \(10\pi^\circ\).

Express in radians the following angles:

9. \(60^\circ\)  10. \(110^\circ 30'\)  11. \(175^\circ 45'\)  12. \(47^\circ 25' 36''\).
13. \(395^\circ\)  14. \(60^\circ\)  15. \(110^\circ 30'\)  16. \(345^\circ 25' 36''.\)

17. The difference between the two acute angles of a right-angled triangle is \(\frac{2}{5}\) \(\pi\) radians; express the angles in degrees.
18. One angle of a triangle is \( \frac{2}{3} \) grades and another is \( \frac{3}{2} \) grades, whilst the third is \( \frac{\pi x}{75} \) radians; express them all in degrees.

19. The circular measure of two angles of a triangle are respectively \( \frac{1}{2} \) and \( \frac{1}{3} \); what is the number of degrees in the third angle?

20. The angles of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest as 60 to \( \pi \); find the angles in degrees.

21. The angles of a triangle are in A.P. and the number of radians in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.

22. Find the magnitude, in radians and degrees, of the interior angle of (1) a regular pentagon, (2) a regular heptagon, (3) a regular octagon, (4) a regular duodecagon, and (5) a regular polygon of 17 sides.

23. The angle in one regular polygon is to that in another as 3 : 2; also the number of sides in the first is twice that in the second; how many sides have the polygons?

24. The number of sides in two regular polygons are as 5 : 4, and the difference between their angles is 9°; find the number of sides in the polygons.

25. Find two regular polygons such that the number of their sides may be as 3 to 4 and the number of degrees in an angle of the first to the number of grades in an angle of the second as 4 to 5.

26. The angles of a quadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

27. Find in radians, degrees, and grades the angle between the hour-hand and the minute-hand of a clock at (1) half-past three, (2) twenty minutes to six, (3) a quarter past eleven.

28. Find the times (1) between four and five o'clock when the angle between the minute-hand and the hour-hand is 78°, (2) between seven and eight o'clock when this angle is 54°.

21. **Theorem.** The number of radians in any angle whatever is equal to a fraction, whose numerator is the arc which the angle subtends at the centre of any circle, and whose denominator is the radius of that circle.
MEASUREMENT OF ANY ANGLE IN RADIANS.

Let $AOP$ be the angle which has been described by a line starting from $OA$ and revolving into the position $OP$.

With centre $O$ and any radius describe a circle cutting $OA$ and $OP$ in the points $A$ and $P$.

Let the angle $AOB$ be a radian, so that the arc $AB$ is equal to the radius $OA$.

By Euc. vi. 33, we have

$$\frac{\angle AOP}{\text{A Radian}} = \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{\text{arc } AP}{\text{Radius}},$$

so that $\angle AOP = \frac{\text{arc } AP}{\text{Radius}}$ of a Radian.

Hence the theorem is proved.

22. Ex. 1. Find the angle subtended at the centre of a circle of radius 3 feet by an arc of length 1 foot.

The number of radians in the angle $= \frac{\text{arc}}{\text{radius}} = \frac{1}{3}$.

Hence the angle

$$= \frac{1}{3} \text{ radian} = \frac{1}{3} \cdot \frac{2}{\pi} \text{ right angle} = \frac{2}{3\pi} \times 90^\circ = \frac{60^\circ}{\pi} = 19\frac{11}{11}^\circ,$$

taking $\pi$ equal to $\frac{22}{7}$.

Ex. 2. In a circle of 5 feet radius what is the length of the arc which subtends an angle of $33^\circ 15'$ at the centre?

If $x$ feet be the required length, we have

$$\frac{x}{5} = \text{number of radians in } 33^\circ 15'$$

$$= \frac{33\frac{1}{4}}{180} \pi \quad (\text{Art. 19}).$$

$$= \frac{183}{720} \pi.$$

$$\therefore x = \frac{183}{144} \pi \text{ feet} = \frac{183}{144} \times \frac{22}{7} \text{ feet nearly}$$

$$= 2\frac{13}{4} \text{ feet nearly.}$$
EX. 3. Assuming the average distance of the earth from the sun to be 92500000 miles, and the angle subtended by the sun at the eye of a person on the earth to be 32', find the sun's diameter.

Let \( D \) be the diameter of the sun in miles.

The angle subtended by the sun being very small, its diameter is very approximately equal to a small arc of a circle whose centre is the eye of the observer. Also the sun subtends an angle of 32' at the centre of this circle.

Hence, by Art. 21, we have

\[
\frac{D}{92500000} = \text{the number of radians in 32'}
\]

\[
= \text{the number of radians in } \frac{8}{15}
\]

\[
= \frac{8}{15} \times \frac{\pi}{180} = \frac{2\pi}{675}
\]

\[
\therefore D = \frac{185000000}{675} \pi \text{ miles}
\]

\[
= \frac{185000000}{675} \times \frac{22}{7} \text{ miles approximately}
\]

\[
= \text{about 862000 miles.}
\]

EX. 4. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5' at his eye, find what is the height of the letters that he can read at a distance (1) of 12 feet, and (2) of a quarter of a mile.

Let \( x \) be the required height in feet.

In the first case, \( x \) is very nearly equal to the arc of a circle, of radius 12 feet, which subtends an angle of 5' at its centre.

Hence

\[
\frac{x}{12} = \text{number of radians in 5'}
\]

\[
= \frac{1}{12} \times \frac{\pi}{180}
\]

\[
\therefore x = \frac{\pi}{180} \text{ feet} = \frac{1}{180} \times \frac{22}{7} \text{ feet nearly}
\]

\[
= \frac{1}{15} \times \frac{22}{7} \text{ inches} = \text{about } \frac{1}{6} \text{ inch.}
\]
MEASUREMENT OF ANY ANGLE IN RADIANS.

In the second case, the height $y$ is given by

$$\frac{y}{440 \times 3} = \text{number of radians in } \mathcal{F}$$

$$= \frac{1}{12} \times \frac{\pi}{180},$$

so that

$$y = \frac{11}{18} \pi = \frac{11}{18} \times \frac{22}{7} \text{ feet nearly}$$

$$= \text{about 23 inches.}$$

EXAMPLES. IV.

\[ Assume \pi = 3.14159 \ldots \text{ and } \frac{1}{\pi} = 0.31831. \]

1. Find the number of degrees subtended at the centre of a circle by an arc whose length is .357 times the radius.

2. Express in radians and degrees the angle subtended at the centre of a circle by an arc whose length is 15 feet, the radius of the circle being 25 feet.

3. The value of the divisions on the outer rim of a graduated circle is 5' and the distance between successive graduations is .1 inch. Find the radius of the circle.

4. The diameter of a graduated circle is 6 feet and the graduations on its rim are 5' apart; find the distance from one graduation to another.

5. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by 1° 10' may be half-an-inch.

6. Taking the radius of the earth as 4000 miles, find the difference in latitude of two places, one of which is 100 miles north of the other.

7. Assuming the earth to be a sphere and the distance between two parallels of latitude, which subtends an angle of 1° at the earth's centre, to be 69$$\frac{1}{4}$$ miles, find the radius of the earth.

8. The radius of a certain circle is 3 feet; find approximately the length of an arc of this circle, if the length of the chord of the arc be 8 feet also.

9. What is the ratio of the radii of two circles at the centre of which two arcs of the same length subtend angles of 60° and 75°?

10. If an arc, of length 10 feet, on a circle of 8 feet diameter subtend at the centre an angle of 143° 14' 22''; find the value of $\pi$ to 4 places of decimals.

L. T.
11. If the circumference of a circle be divided into 5 parts which are in A.P., and if the greatest part be 6 times the least, find in radians the magnitudes of the angles that the parts subtend at the centre of the circle.

12. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees, minutes, and seconds.

13. At what distance does a man, whose height is 6 feet, subtend an angle of 10′?

14. Find the length which at a distance of one mile will subtend an angle of 1′ at the eye.

15. Find approximately the distance at which a globe, 5½ inches in diameter, will subtend an angle of 6′.

16. Find approximately the distance of a tower whose height is 51 feet and which subtends at the eye an angle of 5½′.

17. A church spire, whose height is known to be 100 feet, subtends an angle of 9′ at the eye; find approximately its distance.

18. Find approximately in minutes the inclination to the horizon of an incline which rises 3½ feet in 210 yards.

19. The radius of the earth being taken to be 3960 miles, and the distance of the moon from the earth being 60 times the radius of the earth, find approximately the radius of the moon which subtends at the earth an angle of 16′.

20. When the moon is setting at any given place, the angle that is subtended at its centre by the radius of the earth passing through the given place is 57′. If the earth's radius be 3960 miles, find approximately the distance of the moon.

21. Prove that the distance of the sun is about 81 million geographical miles, taking that the angle which the earth's radius subtends at the distance of the sun is 57′′, and that a geographical mile subtends 1′ at the earth's centre. Find also the circumference and diameter of the earth in geographical miles.

22. The radius of the earth's orbit, which is about 92700000 miles, subtends at the star Sirius an angle of about 4′; find roughly the distance of Sirius.
CHAPTER II.

TRIGONOMETRICAL RATIOS FOR ANGLES LESS THAN
A RIGHT ANGLE.

23. In the present chapter we shall only consider angles which are less than a right angle.

Let a revolving line $OP$ start from $OA$ and revolve into the position $OP$, thus tracing out the angle $AOP$.

In the revolving line take any point $P$ and draw $PM$ perpendicular to the initial line $OA$.

In the triangle $MOP$, $OP$ is the hypotenuse, $PM$ is the perpendicular, and $OM$ is the base.

The trigonometrical ratios, or functions, of the angle $AOP$ are defined as follows:

\[
\frac{MP}{OP}, \text{ i.e. } \frac{\text{Perp.}}{\text{Hyp.}}, \text{ is called the Sine of the angle } AOP; \\
\frac{OM}{OP}, \text{ i.e. } \frac{\text{Base}}{\text{Hyp.}}, \text{ " } \text{ " Cosine } \text{ " } \text{ " } \\
\frac{MP}{OM}, \text{ i.e. } \frac{\text{Perp.}}{\text{Base}}, \text{ " } \text{ " Tangent } \text{ " } \text{ " } \\
\frac{OM}{MP}, \text{ i.e. } \frac{\text{Base}}{\text{Perp.}}, \text{ " } \text{ " Cotangent } \text{ " } \text{ " } \\
\frac{OP}{MP}, \text{ i.e. } \frac{\text{Hyp.}}{\text{Perp.}}, \text{ " } \text{ " Cosecant } \text{ " } \text{ " } \\
\frac{OP}{OM}, \text{ i.e. } \frac{\text{Hyp.}}{\text{Base}}, \text{ " } \text{ " Secant } \text{ " } \text{ " } \\
\]
The quantity by which the cosine falls short of unity, i.e. \(1 - \cos AOP\), is called the **Versed Sine** of \(AOP\); also the quantity \(1 - \sin AOP\), by which the sine falls short of unity, is called the **Coversed Sine** of \(AOP\).

24. It will be noted that the trigonometrical ratios are all numbers.

The names of these eight ratios are written, for brevity, \(\sin AOP\), \(\cos AOP\), \(\tan AOP\), \(\cot AOP\), \(\cosec AOP\), \(\sec AOP\), \(\text{vers } AOP\), and \(\text{covers } AOP\) respectively.

The two latter ratios are seldom used.

25. It will be noticed, from the definitions, that the **cosecant** is the reciprocal of the **sine**, so that

\[
\cosec AOP = \frac{1}{\sin AOP}.
\]

So the **secant** is the reciprocal of the **cosine**, i.e.

\[
\sec AOP = \frac{1}{\cos AOP},
\]

and the **cotangent** is the reciprocal of the **tangent**, i.e.

\[
\cot AOP = \frac{1}{\tan AOP}.
\]

26. **To shew that the trigonometrical ratios are always the same for the same angle.**

We have to shew that, if in the revolving line \(OP\) any other point \(P'\) be taken and \(P'M'\) be drawn perpendicular to \(OA\), the ratios derived from the triangle...
$OP'M'$ are the same as those derived from the triangle $OPM$.

In the two triangles, the angle at $O$ is common, and the angles at $M$ and $M'$ are both right angles and therefore equal.

Hence the two triangles are equiangular and therefore, by Euc. VI. 4, we have $\frac{MP}{OP} = \frac{M'P'}{OP'}$, i.e. the sine of the angle $AOP$ is the same whatever point we take on the revolving line.

Since, by the same proposition, we have

$$\frac{OM}{OP} = \frac{OM'}{OP'} \text{ and } \frac{MP}{OM} = \frac{M'P'}{OM'}$$

it follows that the cosine and tangent are the same whatever point be taken on the revolving line. Similarly for the other ratios.

If $OA$ be considered as the revolving line, and in it be taken any point $P''$ and $P'M''$ be drawn perpendicular to $OP$, the functions as derived from the triangle $OP''M''$ will have the same values as before.

For, since in the two triangles $OPM$ and $OP''M''$, the two angles $P''OM''$ and $OM''P''$ are respectively equal to $POM$ and $OMP$, these two triangles are equiangular and therefore similar, and we have

$$\frac{M''P''}{OP''} = \frac{MP}{OP}, \text{ and } \frac{OM''}{OP''} = \frac{OM}{OP}.$$

27. **Fundamental relations between the trigonometrical ratios of an angle.**

We shall find that if one of the trigonometrical ratios of an angle be known, the numerical magnitude of each of the others is known also.

Let the angle $AOP$ (Fig., Art. 23) be denoted by $\theta$ [pronounced “Theta”].

In the triangle $MOP$ we have, by Euc. I. 47,

$$MP^2 + OM^2 = OP^2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1).$$
Hence, dividing by \( OP^2 \), we have
\[
\left( \frac{MP}{OP} \right)^2 + \left( \frac{OM}{OP} \right)^2 = 1,
\]
i.e.
\[
(\sin \theta)^2 + (\cos \theta)^2 = 1.
\]

The quantity \((\sin \theta)^2\) is always written \(\sin^2 \theta\), and so for
the other ratios.

Hence this relation is
\[
\sin^2 \theta + \cos^2 \theta = 1 \hspace{1cm} (2).
\]

Again, dividing both sides of equation (1) by \( OM^2 \), we have
\[
\left( \frac{MP}{OM} \right)^2 + 1 = \left( \frac{OP}{OM} \right)^2,
\]
i.e.
\[
(\tan \theta)^2 + 1 = (\sec \theta)^2,
\]
so that
\[
\sec^2 \theta = 1 + \tan^2 \theta \hspace{1cm} (3).
\]

Again, dividing equations (1) by \( MP^2 \), we have
\[
1 + \left( \frac{OM}{MP} \right)^2 = \left( \frac{OP}{MP} \right)^2,
\]
i.e.
\[
1 + (\cot \theta)^2 = (\cosec \theta)^2,
\]
so that
\[
\cosec^2 \theta = 1 + \cot^2 \theta \hspace{1cm} (4).
\]

Also, since \( \sin \theta = \frac{MP}{OP} \) and \( \cos \theta = \frac{OM}{OP} \),
we have
\[
\frac{\sin \theta}{\cos \theta} = \frac{MP}{OP} \div \frac{OM}{OP} = \frac{MP}{OM} = \tan \theta.
\]
Hence
\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \hspace{1cm} (5).
\]

Similarly
\[
\cot \theta = \frac{\cos \theta}{\sin \theta} \hspace{1cm} (6).
\]
28. **Ex. 1.** Prove that \( \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \csc A - \cot A \).

We have

\[
\sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\left(\frac{1 - \cos A}{1 - \cos^2 A}\right)} = \frac{1 - \cos A}{\sqrt{1 - \cos^2 A}} = \frac{1 - \cos A}{\sin A},
\]

by relation (2) of the last article,

\[
= \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \csc A - \cot A.
\]

**Ex. 2.** Prove that

\[ \sqrt{\sec^2 A + \csc^2 A} = \tan A + \cot A. \]

We have seen that \( \sec^2 A = 1 + \tan^2 A \),

and \( \csc^2 A = 1 + \cot^2 A \).

\[
\therefore \sec^2 A + \csc^2 A = \tan^2 A + 2 + \cot^2 A = \tan^2 A + 2 \tan A \cot A + \cot^2 A = (\tan A + \cot A)^2,
\]

so that \( \sqrt{\sec^2 A + \csc^2 A} = \tan A + \cot A. \)

**Ex. 3.** Prove that

\[(\csc A - \sin A) (\sec A - \cos A) (\tan A + \cot A) = 1.\]

The given expression

\[
= \left(\frac{1}{\sin A} - \sin A\right) \left(\frac{1}{\cos A} - \cos A\right) \left(\frac{\sin A + \cos A}{\cos A}\right)
\]

\[
= \frac{1 - \sin^2 A}{\sin A} \cdot \frac{1 - \cos^2 A}{\cos A} \cdot \frac{\sin^3 A + \cos^3 A}{\sin A \cos A}
\]

\[
= \frac{\cos^3 A \cdot \sin^2 A \cdot 1}{\sin A \cdot \cos A \cdot \sin A \cos A}
\]

\[= 1.\]
EXAMPLES. V.

Prove the following statements.

1. \( \cos^4 A - \sin^4 A + 1 = 2 \cos^2 A. \)

2. \( (\sin A + \cos A)(1 - \sin A \cos A) = \sin^2 A + \cos^2 A. \)

3. \( \frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \csc A. \)

4. \( \cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A. \)

5. \( \sqrt{\frac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A. \)

6. \( \frac{\csc A}{\csc A - 1} + \frac{\csc A}{\csc A + 1} = 2 \sec^2 A. \)

7. \( \frac{\csc A}{\cot A + \tan A} = \cos A. \)

8. \( (\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A. \)

9. \( \frac{1}{\cot A + \tan A} = \sin A \cos A. \)

10. \( \frac{1}{\sec A - \tan A} = \sec A + \tan A. \)

11. \( \frac{1 - \tan A}{1 + \tan A} = \frac{\cot A - 1}{\cot A + 1}. \)

12. \( \frac{1 + \tan^2 A}{1 + \cot^2 A} = \frac{\sin^2 A}{\cos^2 A}. \)

13. \( \frac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2 \sec A \tan A + 2 \tan^2 A. \)

14. \( \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \sec A \csc A + 1. \)

15. \( \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A. \)

16. \( (\sin A + \cos A)(\cot A + \tan A) = \sec A + \csc A. \)

17. \( \sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A. \)
18. \( \cot^4 A + \cot^2 A = \cosec^4 A - \cosec^2 A. \)

19. \( \sqrt{\cosec^2 A - 1} = \cos A \cosec A. \)

20. \( \sec^2 A \cosec^2 A = \tan^2 A + \cot^2 A + 2. \)

21. \( \tan^2 A - \sin^2 A = \sin^4 A \sec^2 A. \)

22. \( (1 + \cot A - \cosec A) (1 + \tan A + \sec A) = 2. \)

23. \( \frac{1}{\cosec A - \cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{\cosec A + \cot A}. \)

24. \( \frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A - \cos A}{\cot A \cos A}. \)

25. \( \cot A + \tan B = \cot A \tan B. \)

26. \( \left( \frac{1}{\sec^2 a - \cos^2 a} + \frac{1}{\cosec^2 a - \sin^2 a} \right) \cos^2 a \sin^2 a = \frac{1 - \cos^2 a \sin^2 a}{2 + \cos^2 a \sin^2 a}. \)

27. \( \sin^8 A - \cos^8 A = (\sin^3 A - \cos^3 A) (1 - 3 \sin^2 A \cos^2 A). \)

28. \( \frac{\cos A \cosec A - \sin A \sec A}{\cos A + \sin A} = \cosec A - \sec A. \)

29. \( \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}. \)

30. \( (\tan \alpha + \cosec \beta)^2 - (\cot \beta - \sec \alpha)^2 = 2 \tan \alpha \cot \beta \cosec \alpha \sec \beta. \)

31. \( 2 \sec^2 a - \sec^4 a - 2 \cosec^3 a + \cosec^4 a = \cot^4 a - \tan^4 a. \)

32. \( (\sin a + \cosec a)^2 + (\cos a + \sec a)^2 = \tan^2 a + \cot^2 a + 7. \)

33. \( (\cosec A + \cot A) \covers A - (\sec A + \tan A) \vers A = (\cosec A - \sec A) (2 - \vers A \covers A). \)

34. \( (1 + \cot A + \tan A) (\sin A - \cos A) = \frac{\sec A}{\cosec^2 A} - \frac{\cosec A}{\sec^2 A}. \)

35. \( 2 \versin A + \cos^2 A = 1 + \versin^2 A. \)

29. **Limits to the values of the trigonometrical ratios.**

From equation (2) of Art. 27, we have

\[ \sin^2 \theta + \cos^2 \theta = 1. \]
Now \( \sin^2 \theta \) and \( \cos^2 \theta \), being both squares, are both necessarily positive. Hence, since their sum is unity, neither of them can be greater than unity.

[For if one of them, say \( \sin^2 \theta \), were greater than unity, the other, \( \cos^2 \theta \), would have to be negative, which is impossible.]

Hence neither the sine nor the cosine can be numerically greater than unity.

Since \( \sin \theta \) cannot be greater than unity, therefore cosec \( \theta \), which equals \( \frac{1}{\sin \theta} \), cannot be numerically less than unity.

So sec \( \theta \), which equals \( \frac{1}{\cos \theta} \), cannot be numerically less than unity.

30. The foregoing results follow easily from the figure of Art. 23.

For, whatever be the value of the angle \( AOP \), neither the side \( OM \) nor the side \( MP \) is ever greater than \( OP \).

Since \( MP \) is never greater than \( OP \), the ratio \( \frac{MP}{OP} \) is never greater than unity, so that the sine of an angle is never greater than unity.

Also, since \( OM \) is never greater than \( OP \), the ratio \( \frac{OM}{OP} \) is never greater than unity, \( i.e. \) the cosine is never greater than unity.

31. We can express the trigonometrical ratios of an angle in terms of any one of them.
The simplest method of procedure is best shewn by examples.

**Ex. 1.** To express all the trigonometrical ratios in terms of the sine.

Let $AOP$ be any angle $\theta$.

Let the length $OP$ be unity and let the corresponding length of $MP$ be $s$.

By Euc. 1. 47, $OM = \sqrt{OP^2 - MP^2} = \sqrt{1 - s^2}$.

Hence

$$\sin \theta = \frac{MP}{OP} = \frac{s}{1} = s,$$

$$\cos \theta = \frac{OM}{OP} = \sqrt{1 - s^2} = \sqrt{1 - \sin^2 \theta},$$

$$\tan \theta = \frac{MP}{OM} = \frac{s}{\sqrt{1 - s^2}} = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}},$$

$$\cot \theta = \frac{OM}{MP} = \frac{\sqrt{1 - s^2}}{s} = \frac{1}{\sin \theta},$$

$$\cosec \theta = \frac{OP}{MP} = \frac{1}{s} = \frac{1}{\sin \theta},$$

and

$$\sec \theta = \frac{OP}{OM} = \frac{1}{\sqrt{1 - s^2}} = \frac{1}{\sqrt{1 - \sin^2 \theta}}.$$
Hence \[ \cot \theta = \frac{OM}{MP} = \frac{x}{1} = x, \]

\[ \sin \theta = \frac{MP}{OP} = \frac{1}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{1 + \cot^2 \theta}}, \]

\[ \cos \theta = \frac{OM}{OP} = \frac{x}{\sqrt{1 + x^2}} = \frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}}, \]

\[ \tan \theta = \frac{MP}{OM} = \frac{1}{x} = \frac{1}{\cot \theta}, \]

\[ \sec \theta = \frac{OP}{OM} = \frac{\sqrt{1 + x^2}}{x} = \frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}, \]

and \[ \cosec \theta = \frac{OP}{MP} = \frac{\sqrt{1 + x^2}}{1} = \sqrt{1 + \cot^2 \theta}. \]

The last five equations give what is required.

It will be noticed that, in each case, the denominator of the fraction which defines the trigonometrical ratio was taken equal to unity. For example, the sine is \( \frac{MP}{OP} \), and hence in Ex. 1 the denominator \( OP \) is taken equal to unity.

The cotangent is \( \frac{OM}{MP} \), and hence in Ex. 2 the side \( MP \) is taken equal to unity.

Similarly suppose we had to express the other ratios in terms of the cosine, we should, since the cosine is equal to \( \frac{OM}{OP} \), put \( OP \) equal to unity and \( OM \) equal to \( a \). The working would then be similar to that of Exs. 1 and 2.

In the following examples the sides have numerical values.
**Ex. 3.** If \( \cos \theta \) equal \( \frac{3}{5} \), find the values of the other ratios.

Along the initial line \( OA \) take \( OM \) equal to 3, and erect a perpendicular \( MP \).

Let a line \( OP \), of length 5, revolve round \( O \) until its other end meets this perpendicular in the point \( P \). Then \( AOP \) is the angle \( \theta \).

By Eucl. i. 47, \[ MP = \sqrt{OP^2 - OM^2} = \sqrt{5^2 - 3^2} = 4. \]

Hence clearly

\[ \sin \theta = \frac{4}{5}, \quad \tan \theta = \frac{4}{3}, \quad \cot \theta = \frac{3}{4}, \quad \cosec \theta = \frac{5}{4}, \quad \text{and} \quad \sec \theta = \frac{5}{3}. \]

**Ex. 4.** Supposing \( \theta \) to be an angle whose sine is \( \frac{1}{3} \), to find the numerical magnitude of the other trigonometrical ratios.

Here \( \sin \theta = \frac{1}{3} \), so that the relation (2) of Art. 27 gives

\[ \left( \frac{1}{3} \right)^2 + \cos^2 \theta = 1, \]

i.e.

\[ \cos^2 \theta = 1 - \frac{1}{9} = \frac{8}{9}, \]

i.e.

\[ \cos \theta = \frac{2\sqrt{2}}{3}. \]

Hence

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}, \]

\[ \cot \theta = \frac{1}{\tan \theta} = 2\sqrt{2}, \]

\[ \cosec \theta = \frac{1}{\sin \theta} = 3, \]

\[ \sec \theta = \frac{1}{\cos \theta} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}, \]

\[ \text{vers} \theta = 1 - \cos \theta = 1 - \frac{2\sqrt{2}}{3}, \]

and

\[ \text{overs} \theta = 1 - \sin \theta = 1 - \frac{1}{3} = \frac{2}{3}. \]
32. In the following table is given the result of expressing each trigonometrical ratio in terms of each of the others.

<table>
<thead>
<tr>
<th></th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
<th>$\cot \theta$</th>
<th>$\sec \theta$</th>
<th>$\cosec \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>$\sin \theta$</td>
<td>$\sqrt{1 - \cos^2 \theta}$</td>
<td>$\tan \theta$</td>
<td>$\frac{1}{\sqrt{1 + \cot^2 \theta}}$</td>
<td>$\frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}$</td>
<td>$\frac{1}{\cosec \theta}$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$\sqrt{1 - \sin^2 \theta}$</td>
<td>$\cos \theta$</td>
<td>$\frac{1}{\sqrt{1 + \tan^2 \theta}}$</td>
<td>$\cot \theta$</td>
<td>$\frac{1}{\sec \theta}$</td>
<td>$\frac{\sqrt{\cosec^2 \theta - 1}}{\cosec \theta}$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>$\frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$</td>
<td>$\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$</td>
<td>$\tan \theta$</td>
<td>$\frac{1}{\cot \theta}$</td>
<td>$\frac{\sqrt{\sec^2 \theta - 1}}{\cosec \theta}$</td>
<td>$\frac{1}{\sqrt{\cosec^2 \theta - 1}}$</td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td>$\frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta}$</td>
<td>$\frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}}$</td>
<td>$\frac{1}{\tan \theta}$</td>
<td>$\cot \theta$</td>
<td>$\frac{1}{\sqrt{\sec^2 \theta - 1}}$</td>
<td>$\frac{\sqrt{\cosec^2 \theta - 1}}{\cosec \theta}$</td>
</tr>
<tr>
<td>$\sec \theta$</td>
<td>$\frac{1}{\sqrt{1 - \sin^2 \theta}}$</td>
<td>$\frac{1}{\cos \theta}$</td>
<td>$\sqrt{1 + \tan^2 \theta}$</td>
<td>$\frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}$</td>
<td>$\sec \theta$</td>
<td>$\frac{\cosec \theta}{\sqrt{\cosec^2 \theta - 1}}$</td>
</tr>
<tr>
<td>$\cosec \theta$</td>
<td>$\frac{1}{\sin \theta}$</td>
<td>$\frac{1}{\sqrt{1 - \cos^2 \theta}}$</td>
<td>$\frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}$</td>
<td>$\frac{\sqrt{1 + \cot^2 \theta}}{\sec \theta}$</td>
<td>$\frac{\sec \theta}{\sqrt{\sec^2 \theta - 1}}$</td>
<td>$\cosec \theta$</td>
</tr>
</tbody>
</table>
EXAMPLES. VI.

1. Express all the other trigonometrical ratios in terms of the cosine.
2. Express all the ratios in terms of the tangent.
3. Express all the ratios in terms of the cosecant.
4. Express all the ratios in terms of the secant.
5. The sine of a certain angle is $\frac{1}{4}$; find the numerical values of the other trigonometrical ratios of this angle.

6. If $\sin \theta = \frac{12}{13}$, find $\tan \theta$ and $versin \theta$.

7. If $\sin A = \frac{11}{61}$, find $\tan A$, $\cos A$, and $\sec A$.

8. If $\cos \theta = \frac{4}{5}$, find $\sin \theta$ and $\cot \theta$.

9. If $\cos A = \frac{9}{41}$, find $\tan A$ and $\csc A$.

10. If $\tan \theta = \frac{3}{4}$, find the sine, cosine, versine and cosecant of $\theta$.

11. If $\tan \theta = \frac{1}{\sqrt{7}}$, find the value of $\frac{\csc^2 \theta - \sec^2 \theta}{\csc^2 \theta + \sec^2 \theta}$.

12. If $\cot \theta = \frac{15}{8}$, find $\cos \theta$ and $\csc \theta$.

13. If $\sec A = \frac{3}{2}$, find $\tan A$ and $\csc A$.

14. If $2 \sin \theta = 2 - \cos \theta$, find $\sin \theta$.

15. If $8 \sin \theta = 4 + \cos \theta$, find $\sin \theta$.

16. If $\tan \theta + \sec \theta = 1.5$, find $\sin \theta$.

17. If $\cot \theta + \csc \theta = 5$, find $\cos \theta$.

18. If $3 \sec^4 \theta + 8 = 10 \sec^2 \theta$, find the values of $\tan \theta$.

19. If $\tan^2 \theta + \sec \theta = 5$, find $\cos \theta$.

20. If $\tan \theta + \cot \theta = 2$, find $\sin \theta$.

21. If $\sec^2 \theta = 2 + 2 \tan \theta$, find $\tan \theta$.

22. If $\tan \theta = \frac{2x(x+1)}{2x+1}$, find $\sin \theta$ and $\cos \theta$. 
Values of the trigonometrical ratios in some useful cases.

33. Angle of 45°
Let the angle \( \angle AOP \) traced out be 45°.

Then, since the three angles of a triangle are together equal to two right angles,
\[
\angle OPM = 180° - \angle POM - \angle PMO
\]
\[
= 180° - 45° - 90° = 45° = \angle POM.
\]
\[\therefore \ OM = MP.\]

If \( OP \) be called 2\( a \), we then have
\[
4a^2 = OP^2 = OM^2 + MP^2 = 2 \cdot OM^2,
\]
so that
\[OM = a/\sqrt{2}.\]
\[\therefore \ \sin 45° = \frac{MP}{OP} = \frac{a/\sqrt{2}}{2a} = \frac{1}{\sqrt{2}},\]
\[\cos 45° = \frac{OM}{OP} = \frac{a/\sqrt{2}}{2a} = \frac{1}{\sqrt{2}},\]
and
\[\tan 45° = 1.\]

34. Angle of 30°.
Let the angle \( \angle AOP \) traced out be 30°.

Produce \( PM \) to \( P' \) making \( MP' \) equal to \( PM \).

The two triangles \( OMP \) and \( OMP' \) have their sides \( OM \) and \( MP' \) equal to \( OM \) and \( MP \) and also the contained angles equal.

Therefore \( OP' = OP \), and \( \angle OP'P = \angle OPP' = 60° \), so that the triangle \( P'OP \) is equilateral.
Hence, if $OP$ be called $2a$, we have

$$MP = \frac{1}{2}PP = \frac{1}{2}OP = a.$$  

Also

$$OM = \sqrt{OP^2 - MP^2} = \sqrt{4a^2 - a^2} = a\sqrt{3}.$$  

$$\therefore \sin 30^\circ = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$$

and

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}.$$  

35. **Angle of 60°.**

Let the angle $AOP$ traced out be $60°$.

Take a point $N$ on $OA$, so that

$$MN = OM = a \text{ (say).}$$

The two triangles $OMP$ and $NMP$ have now the sides $OM$ and $MP$ equal to $NM$ and $MP$ respectively, and the included angles equal, so that the triangles are equal.

$$\therefore PN = OP, \text{ and } \angle PNM = \angle POM = 60°.$$  

The triangle $OPN$ is therefore equilateral, and hence

$$OP = ON = 2OM = 2a.$$  

$$\therefore MP = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}.a.$$  

L. T.
Hence $\sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$,

$\cos 60^\circ = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$,

and $\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$.

36. **Angle of 0°.**

Let the revolving line $OP$ have turned through a very small angle, so that the angle $MOP$ is very small.

The magnitude of $MP$ is then very small, and initially, before $OP$ had turned through an angle large enough to be perceived, the quantity $MP$ was smaller than any quantity we could assign, i.e. was what we denote by 0.

Also, in this case, the two points $M$ and $P$ very nearly coincide, and the smaller the angle $AOP$ the more nearly do they coincide.

Hence, when the angle $AOP$ is actually zero, the two lengths $OM$ and $OP$ are equal and $MP$ is zero.

Hence $\sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0$,

$\cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1$,

and $\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$.

Also $\cot 0^\circ = \frac{OM}{MP}$ when $M$ and $P$ coincide
= the ratio of a finite quantity to something infinitely small
= a quantity which is infinitely great.

Such a quantity is usually denoted by the symbol $\infty$. 
Hence \[ \cot 0^\circ = \infty. \]

Similarly \[ \csc 0^\circ = \frac{OP}{MP} = \infty \text{ also.} \]

And \[ \sec 0^\circ = \frac{OP}{OM} = 1. \]

37. Angle of 90°.

Let the angle \( AOP \) be very nearly, but not quite, a right angle.

When \( OP \) has actually described a right angle, the point \( M \) coincides with \( O \), so that then \( OM \) is zero and \( OP \) and \( MP \) are equal.

Hence \[ \sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1, \]

\[ \cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0, \]

\[ \tan 90^\circ = \frac{MP}{OM} = \frac{\text{a finite quantity}}{\text{an infinitely small quantity}} = \text{a number infinitely large} = \infty, \]

\[ \cot 90^\circ = \frac{OM}{MP} = \frac{0}{MP} = 0, \]

\[ \sec 90^\circ = \frac{OP}{OM} = \infty, \text{ as in the case of the tangent,} \]

and \[ \cosec 90^\circ = \frac{OP}{MP} = \frac{OP}{OP} = 1. \]

38. Complementary Angles. Def. Two angles are said to be complementary when their sum is equal to a right angle. Thus any angle \( \theta \) and the angle \( 90^\circ - \theta \) are complementary.
39. To find the relations between the trigonometrical ratios of two complementary angles.

Let the revolving line, starting from \( OA \), trace out any acute angle \( AOP \), equal to \( \theta \). From any point \( P \) on it draw \( PM \) perpendicular to \( OA \).

Since the three angles of a triangle are together equal to two right angles, and since \( OMP \) is a right angle, the sum of the two angles \( MOP \) and \( OPM \) is a right angle.

They are therefore complementary and

\[ \angle OPM = 90^\circ - \theta. \]

[When the angle \( OPM \) is considered, the line \( PM \) is the "base" and \( MO \) is the "perpendicular."]

We then have

\[
\sin (90^\circ - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos \theta,
\]

\[
\cos (90^\circ - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin \theta,
\]

\[
\tan (90^\circ - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot \theta,
\]

\[
\cot (90^\circ - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan \theta,
\]

\[
\cosec (90^\circ - \theta) = \cosec MPO = \frac{PO}{MO} = \sec AOP = \sec \theta,
\]

and \( \sec (90^\circ - \theta) = \sec MPO = \frac{PO}{PM} = \cosec AOP = \cosec \theta. \)
Hence we observe that
the Sine of any angle = the Cosine of its complement,
the Tangent of any angle = the Cotangent of its complement,
and the Secant of an angle = the Cosecant of its complement.

From this is apparent what is the derivation of the names Cosine, Cotangent, and Cosecant.

40. The student is advised before proceeding any further to make himself quite familiar with the following table. [For an extension of this table, see Art. 76.]

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>Cosine</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>Tangent</td>
<td>0</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Cotangent</td>
<td>$\infty$</td>
<td>$\sqrt{3}$</td>
<td>1</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>0</td>
</tr>
<tr>
<td>Cosecant</td>
<td>$\infty$</td>
<td>2</td>
<td>$\sqrt{2}$</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>1</td>
</tr>
<tr>
<td>Secant</td>
<td>1</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>$\sqrt{2}$</td>
<td>2</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

If the student commits accurately to memory the portion of the above table included between the thick lines, he should be able to easily reproduce the rest.
For
(1) the sines of 60° and 90° are respectively the
cosines of 30° and 0°. \hspace{1cm} (Art. 39.)
(2) the cosines of 60° and 90° are respectively the
sines of 30° and 0°. \hspace{1cm} (Art. 39.)
Hence the second and third lines are known.
(3) The tangent of any angle is the result of dividing
the sine by the cosine.
Hence any quantity in the fourth line is obtained by
dividing the corresponding quantity in the second line by
the corresponding quantity in the third line.
(4) The cotangent of any angle is the reciprocal of
the tangent, so that the quantities in the fifth row are the
reciprocals of the quantities in the fourth row.
(5) Since cosec $\theta = \frac{1}{\sin \theta}$, the sixth row is obtained
by inverting the corresponding quantities in the second
row.
(6) Since sec $\theta = \frac{1}{\cos \theta}$, the seventh row is similarly
obtained from the third row.

**EXAMPLES. VII**

1. If $A = 30^\circ$, verify that
   (1) $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1$,
   (2) $\sin 2A = 2 \sin A \cos A$,
   (3) $\cos 3A = 4 \cos^3 A - 3 \cos A$,
   (4) $\sin 3A = 3 \sin A - 4 \sin^3 A$,
and (5) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$. 
2. If $\theta = 45^\circ$, verify that
   
   (1) $\sin 2\theta = 2 \sin \theta \cos \theta$,
   
   (2) $\cos 2\theta = 1 - 2 \sin^2 \theta$,
   
   and (3) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$.

   Verify that

3. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ = \frac{3}{2}$.

4. $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ = 4\frac{1}{2}$.

5. $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = 1$.

6. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ = -\frac{\sqrt{3} - 1}{2\sqrt{2}}$.

7. $\frac{1}{2} \cot^2 30^\circ + 3 \sin^2 60^\circ - 2 \cosec^2 60^\circ - \frac{1}{2} \tan^2 30^\circ = 3\frac{1}{2}$.

8. $\cosec^2 45^\circ \cdot \sec^2 30^\circ \cdot \sin^3 90^\circ \cdot \cos 60^\circ = 1\frac{1}{2}$.

9. $4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^3 30^\circ = \frac{1}{4}$. 

CHAPTER III.

SIMPLE PROBLEMS IN HEIGHTS AND DISTANCES.

41. One of the objects of Trigonometry is to find the distances between points, or the heights of objects, without actually measuring these distances or these heights.

42. Suppose $O$ and $P$ to be two points, $P$ being at a higher level than $O$.

Let $OM$ be a horizontal line drawn through $O$ to meet in $M$ the vertical line drawn through $P$.

The angle $MOP$ is called the **Angle of Elevation** of the point $P$ as seen from $O$.

Draw $PN$ parallel to $MO$, so that $PN$ is the horizontal line passing through $P$. The angle $NPO$ is the **Angle of Depression** of the point $O$ as seen from $P$.

43. Two of the instruments used in practical work are the Theodolite and the Sextant.

The Theodolite is used to measure angles in a vertical plane.

The Theodolite, in its simple form, consists of a telescope attached to a flat piece of wood. This piece of wood is supported by three legs and can be arranged so as to be accurately horizontal.
This table being at $O$ and horizontal, and the telescope being initially pointing in the direction $OM$, the latter can be made to rotate in a vertical plane until it points accurately towards $P$. A graduated scale shows the angle through which it has been turned from the horizontal, i.e. gives us the angle of elevation $MOP$.

Similarly, if the instrument were at $P$, the angle $NPO$ through which the telescope would have to be turned, downward from the horizontal, would give us the angle $NPO$.

The instrument can also be used to measure angles in a horizontal plane.

44. The Sextant is used to find the angle subtended by any two points $D$ and $E$ at a third point $F$. It is an instrument much used on board ships.

Its construction and application are too complicated to be here considered.

45. We shall now solve a few simple examples in heights and distances.

**Ex. 1.** A vertical flagstaff stands on a horizontal plane; from a point distant 150 feet from its foot, the angle of elevation of its top is found to be $30^\circ$; find the height of the flagstaff.

Let $MP$ (Fig. Art. 42) represent the flagstaff and $O$ the point from which the angle of elevation is taken.

Then $OM=150$ feet, and $\angle MOP=30^\circ$.

Since $PMO$ is a right angle, we have

\[
\frac{MP}{OM} = \tan MOP = \tan 30^\circ = \frac{1}{\sqrt{3}} \quad \text{(Art. 34)}.
\]

\[
\therefore \quad MP = \frac{OM}{\sqrt{3}} = \frac{150}{\sqrt{3}} = \frac{150\sqrt{3}}{3} = 50\sqrt{3}.
\]

Now, by extraction of the square root, we have

\[
\sqrt{3} = 1.73205...\text{...}
\]

Hence

\[
MP = 50 \times 1.73205... \text{...feet} = 86.6025... \text{...feet}.
\]

**Ex. 2.** A man wishes to find the height of a church spire which stands on a horizontal plane; at a point on this plane he finds the angle of elevation of the top of the spire to be $45^\circ$; on walking 100 feet toward the tower he finds the corresponding angle of elevation to be $60^\circ$; deduce the height of the tower and also his original distance from the foot of the spire.
Let $P$ be the top of the spire and $A$ and $B$ the two points at which the angles of elevation are taken. Draw $PM$ perpendicular to $AB$ produced and let $MP$ be $x$.

We are given $AB = 100$ feet,

$\angle MAP = 45^\circ$,

and $\angle MBP = 60^\circ$.

We then have

$$\frac{AM}{x} = \cot 45^\circ,$$

and $\frac{BM}{x} = \cot 60^\circ = \frac{1}{\sqrt{3}}$.

Hence $AM = x$, and $BM = \frac{x}{\sqrt{3}}$.

\[
\therefore \quad 100 = AM - BM = x - \frac{x}{\sqrt{3}} = x \frac{\sqrt{3} - 1}{\sqrt{3}}.
\]

\[
= 50 \left(3 + 1\cdot73205\ldots\right) = 236\cdot6\ldots\text{ feet}.
\]

Also $AM = x$, so that both of the required distances are equal to $236\cdot6\ldots\text{ feet}$.

**Ex. 3.** From the top of a cliff, 200 feet high, the angles of depression of the top and bottom of a tower are observed to be $30^\circ$ and $60^\circ$; find the height of the tower.

Let $A$ be the point of observation and $BA$ the height of the cliff and let $CD$ be the tower.

Draw $AE$ horizontally, so that $\angle EAC = 30^\circ$ and $\angle EAD = 60^\circ$.

Let $x$ feet be the height of the tower and produce $DC$ to meet $AE$ in $E$, so that $CE = AB - x = 200 - x$.

Since $\angle ADB = \angle DAE = 60^\circ$ (Euc. i. 29),

$$DB = AB \cot ADB = 200 \cot 60^\circ = \frac{200}{\sqrt{3}}.$$

Also

$$\frac{200 - x}{DB} = \frac{CE}{EA} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$
HEIGjHTS AND DISTANCES.

\[ 200 - x = \frac{DB}{\sqrt{3}} = \frac{200}{3}, \]

so that \[ x = 200 - \frac{200}{3} = 133\frac{1}{3} \text{ feet}. \]

**Ex. 4.** A man observes that at a point due south of a certain tower its angle of elevation is 60°; he then walks 300 feet due west on a horizontal plane and finds that the angle of elevation is 30°; find the height of the tower and his original distance from it.

Let \( P \) be the top, and \( PM \) the height, of the tower; \( A \) the point due south of the tower and \( B \) the point due west of \( A \).

The angles \( PMA, PMB, \) and \( MAB \) are therefore all right angles.

For simplicity, since the triangles \( PAM, PBM, \) and \( ABM \) are in different planes, they are reproduced in the second, third, and fourth figures and drawn to scale.

We are given \( AB = 300 \) feet, \( \angle PAM = 60° \), and \( \angle PBM = 30° \).

Let the height of the tower be \( x \) feet.

From the second figure,

\[
\frac{AM}{x} = \cot 60° = \frac{1}{\sqrt{3}},
\]

so that \[ AM = \frac{x}{\sqrt{3}}. \]

From the third figure,

\[
\frac{BM}{x} = \cot 30° = \sqrt{3},
\]

so that \[ BM = \sqrt{3} \cdot x. \]
From the last figure, we have

\[ BM^2 = AM^2 + AB^2, \]

\[ 3x^2 = \frac{1}{3} x^2 + 300^2. \]

\[ \therefore 8x^2 = 3 \times 300^2. \]

\[ \therefore x = \frac{300 \sqrt{3}}{2 \sqrt{2}} = 150 \cdot \frac{\sqrt{6}}{2} = 75 \times \sqrt{6} \]

\[ = 75 \times 2.44949... = 183.71... \text{ feet.} \]

Also his original distance from the tower

\[ \theta = x \cot 60^\circ = \frac{x}{\sqrt{3}} = 75 \times \sqrt{2} \]

\[ = 75 \times (1.4142...) = 106.065... \text{ feet.} \]

**EXAMPLES. VIII.**

1. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retires 40 feet from the bank he finds the angle to be 30°; find the height of the tree and the breadth of the river.

2. At a certain point the angle of elevation of a tower is found to be such that its cotangent is \( \frac{3}{5} \); on walking 32 feet directly toward the tower its angle of elevation is an angle whose cotangent is \( \frac{2}{5} \). Find the height of the tower.

3. At a point \( A \), the angle of elevation of a tower is found to be such that its tangent is \( \frac{5}{12} \); on walking 240 feet nearer the tower the tangent of the angle of elevation is found to be \( \frac{3}{4} \); what is the height of the tower?

4. Find the height of a chimney when it is found that, on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

5. An observer on the top of a cliff, 200 feet above the sea-level, observes the angles of depression of two ships at anchor to be 45° and 30° respectively; find the distances between the ships if the line joining them points to the base of the cliff.
6. From the top of a cliff an observer finds that the angles of depression of two buoys in the sea are $39^\circ$ and $26^\circ$ respectively; the buoys are 300 yards apart and the line joining them points straight at the foot of the cliff; find the height of the cliff and the distance of the nearest buoy from the foot of the cliff, given that $\cot 26^\circ = 2.0503$, and $\cot 39^\circ = 1.2349$.

7. The upper part of a tree broken over by the wind makes an angle of $30^\circ$ with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet; what was the height of the tree?

8. The horizontal distance between two towers is 60 feet and the angular depression of the top of the first as seen from the top of the second, which is 150 feet high, is $30^\circ$; find the height of the first.

9. The angle of elevation of the top of an unfinished tower at a point distant 120 feet from its base is $45^\circ$; how much higher must the tower be raised so that its angle of elevation at the same point may be $60^\circ$?

10. Two pillars of equal height stand on either side of a roadway which is 100 feet wide; at a point in the roadway between the pillars the elevations of the tops of the pillars are $60^\circ$ and $30^\circ$; find their height and the position of the point.

11. The angle of elevation of the top of a tower is observed to be $60^\circ$; at a point 40 feet above the first point of observation the elevation is found to be $45^\circ$; find the height of the tower and its horizontal distance from the points of observation.

12. At the foot of a mountain the elevation of its summit is found to be $45^\circ$; after ascending one mile towards the mountain up a slope of $30^\circ$ inclination the elevation is found to be $60^\circ$. Find the height of the mountain.

13. What is the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole?

14. The shadow of a tower standing on a level plane is found to be 60 feet longer when the sun's altitude is $30^\circ$ than when it is $45^\circ$. Prove that the height of the tower is $30(1 + \sqrt{3})$ feet.

15. On a straight coast there are three objects $A$, $B$, and $C$ such that $AB = BC = 2$ miles. A vessel approaches $B$ in a line perpendicular to the coast and at a certain point $AC$ is found to subtend an angle of $60^\circ$; after sailing in the same direction for ten minutes $AC$ is found to subtend $120^\circ$; find the rate at which the ship is going.
16. Two flagstaffs stand on a horizontal plane. $A$ and $B$ are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from $A$ are $30^\circ$ and $60^\circ$ and, as seen from $B$, they are $60^\circ$ and $45^\circ$. If the length $AB$ be 30 feet, find the heights of the flagstaffs and the distance between them.

17. $P$ is the top and $Q$ the foot of a tower standing on a horizontal plane. $A$ and $B$ are two points on this plane such that $AB$ is 32 feet and $QAB$ is a right angle. It is found that $\cot P AQ = \frac{2}{5}$ and $\cot P B Q = \frac{3}{5}$; find the height of the tower.

18. A square tower stands upon a horizontal plane. From a point in this plane, from which three of its upper corners are visible, their angular elevations are respectively $45^\circ$, $60^\circ$, and $45^\circ$. Shew that the height of the tower is to the breadth of one of its sides as $\sqrt{6} (\sqrt{5} + 1)$ to 4.

19. A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. An observer on a steamer, sailing due west, first sees the light when he is 5 miles away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

20. A man stands at a point $X$ on the bank $XY$ of a river with straight and parallel banks and observes that the line joining $X$ to a point $Z$ on the opposite bank makes an angle of $30^\circ$ with $XY$. He then goes along the bank a distance of 200 yards to $Y$ and finds that the angle $XYZ$ is $60^\circ$. Find the breadth of the river.

21. A man, walking due north, observes that the elevation of a balloon, which is due east of him and is sailing toward the north-west, is then $60^\circ$; after he has walked 400 yards the balloon is vertically over his head; find its height supposing it to have always remained the same.
CHAPTER IV.

APPLICATION OF ALGEBRAIC SIGNS TO TRIGONOMETRY.

46. Positive and Negative Angles. In Art. 6, in treating of angles of any size, we spoke of the revolving line as if it always revolved in a direction opposite to that in which the hands of a watch revolve, when the watch is held with its face uppermost.

This direction is called counter-clockwise.

When the revolving line turns in this manner it is said to revolve in the positive direction and to trace out a positive angle.

When the line $OP$ revolves in the opposite direction, i.e. in the same direction as the hands of the watch, it is said to revolve in the negative direction and to trace out a negative angle. This negative direction is clockwise.

47. Let the revolving line start from $OA$ and revolve until it reaches a position $OP$, which lies between $OA'$ and $OB'$ and which bisects the angle $A'OB'$.

If it has revolved in the positive direction, it has traced out the positive angle whose measure is $+\ 225^\circ$. 
If it has revolved in the negative direction, it has traced out the negative angle $-135^\circ$.

Again, suppose we only know that the revolving line is in the above position. It may have made one, two, three... complete revolutions and then have described the positive angle $+225^\circ$. Or again, it may have made one, two, three... complete revolutions in the negative direction and then have described the negative angle $-135^\circ$.

In the first case, the angle it has described is either $225^\circ$, or $360^\circ + 225^\circ$, or $2 \times 360^\circ + 225^\circ$, or $3 \times 360^\circ + 225^\circ$ .......i.e. $225^\circ$, or $585^\circ$, or $945^\circ$, or $1305^\circ$....

In the second case, the angle it has described is $-135^\circ$, or $-360^\circ - 135^\circ$, or $-2 \times 360^\circ - 135^\circ$, or $-3 \times 360^\circ - 135^\circ$ ....... i.e. $-135^\circ$, or $-495^\circ$, or $-855^\circ$, or $-1215^\circ$....

48. Positive and Negative Lines. Suppose that a man is told to start from a given milestone on a straight road and to walk 1000 yards along the road and then to stop. Unless we are told the direction in which he started, we do not know his position when he stops. All we know is that he is either at a distance 1000 yards on one side of the milestone or at the same distance on the other side.

In measuring distances along a straight line it is therefore convenient to have a standard direction; this direction is called the positive direction and all distances measured along it are said to be positive. The opposite direction is called the negative direction, and all distances measured along it are said to be negative.

The standard, or positive, directions for lines drawn parallel to the foot of the page is towards the right.
The length \( OA \) is in the positive direction. The length \( OA' \) is in the negative direction. If the magnitude of the distance \( OA \) or \( OA' \) be \( a \), the point \( A \) is at a distance \( +a \) from \( O \) and the point \( A' \) is at a distance \( -a \) from \( O \).

All lines measured to the right have then the positive sign prefixed; all lines to the left have the negative sign prefixed.

If a point start from \( O \) and describe a positive distance \( OA \), and then a distance \( AB \) back again toward \( O \), equal numerically to \( b \), the total distance it has described measured in the positive direction is \( OA + AB \),

\[
i.e. \ +a + (-b), \ i.e. \ a - b.
\]

49. For lines at right angles to \( AA' \), the positive direction is from \( O \) towards the top of the page, i.e. the direction of \( OB \) (Fig. Art. 47). All lines measured from \( O \) towards the foot of the page, i.e. in the direction \( OB' \) are negative.

50. *Trigonometrical ratios for an angle of any magnitude.*

Let \( OA \) be the initial line (drawn in the positive direction) and let \( OA' \) be drawn in the opposite direction to \( OA \).

Let \( BOB' \) be a line at right angles to \( OA \), its positive direction being \( OB \).

Let a revolving line \( OP \) start from \( OA \) and revolving in either direction, positive or negative, trace
out an angle of any magnitude whatever. From a point \( P \) in the revolving line draw \( PM \) perpendicular to \( AOA' \).

[Four positions of the revolving line are given in the figure, one in each of the four quadrants, and the suffixes 1, 2, 3 and 4 are attached to \( P \) for the purpose of distinction.]

We then have the following definitions, which are the same as those given in Art. 23 for the simple case of an acute angle:

\[
\frac{MP}{OP} \text{ is called the Sine of the angle } AOP,
\]

\[
\frac{OM}{OP} \quad \text{Cosine}
\]

\[
\frac{MP}{OM} \quad \text{Tangent}
\]

\[
\frac{OM}{MP} \quad \text{Cotangent}
\]

\[
\frac{OP}{OM} \quad \text{Secant}
\]

\[
\frac{OP}{MP} \quad \text{Cosecant}
\]

The quantities \( 1 - \cos AOP \), and \( 1 - \sin AOP \) are respectively called the **Versed Sine** and the **Covered Sine** of \( AOP \).

**51.** In exactly the same manner as in Art. 27 it may be shewn that, for all values of the angle \( AOP (= \theta) \), we have
\[ \sin^2 \theta + \cos^2 \theta = 1, \]
\[ \frac{\sin \theta}{\cos \theta} = \tan \theta, \]
\[ \sec^2 \theta = 1 + \tan^2 \theta, \]
and \[ \cosec^2 \theta = 1 + \cot^2 \theta. \]

52. Signs of the trigonometrical ratios.

*First quadrant.* Let the revolving line be in the first quadrant, as \( OP_1 \). This revolving line is always positive. Here \( OM_1 \) and \( M_1P_1 \) are both positive, so that all the trigonometrical ratios are then positive.

*Second quadrant.* Let the revolving line be in the second quadrant, as \( OP_2 \). Here \( M_2P_2 \) is positive and \( OM_2 \) is negative.

The sine, being equal to the ratio of a positive quantity to a positive quantity, is therefore positive.

The cosine, being equal to the ratio of a negative quantity to a positive quantity, is therefore negative.

The tangent, being equal to the ratio of a positive quantity to a negative quantity, is therefore negative.

The cotangent is negative.

The cosecant is positive.

The secant is negative.

*Third quadrant.* If the revolving line be, as \( OP_3 \), in the third quadrant, we have both \( M_3P_3 \) and \( OM_3 \) negative. The sine is therefore negative.

The cosine is negative.

The tangent is positive.

The cotangent is positive.

The cosecant is negative.

The secant is negative.
Fourth quadrant. Let the revolving line be in the fourth quadrant, as \( OP_4 \). Here \( M_4P_4 \) is negative and \( OM_4 \) is positive.

The sine is therefore negative.
The cosine is positive.
The tangent is negative.
The cotangent is negative.
The cosecant is negative.
The secant is positive.

The annexed table shews the signs of the trigonometrical ratios according to the quadrant in which lies the revolving line, which bounds the angle considered.

\[
\begin{array}{c|c|c|}
\text{sin} & +& \text{sin} & + \\
\text{cos} & - & \text{cos} & + \\
\text{tan} & - & \text{tan} & + \\
\text{cot} & - & \text{cot} & + \\
\text{cosec} & + & \text{cosec} & + \\
\text{sec} & - & \text{sec} & + \\
\end{array}
\]

53. Tracing of the changes in the sign and magnitude of the trigonometrical ratios of an angle, as the angle increases from 0° to 360°.

Let the revolving line \( OP \) be of constant length \( a \).
When it coincides with \(OA\), the length \(OM_1\) is equal to \(a\) and, when it coincides with \(OB\), the point \(M_1\) coincides with \(O\) and \(OM_1\) vanishes. Also, as the revolving line turns from \(OA\) to \(OB\), the distance \(OM_1\) decreases from \(a\) to zero.

Whilst the revolving line is in the second quadrant and is revolving from \(OB\) to \(OA'\), the distance \(OM_2\) is negative and increases numerically from 0 to \(a\) [i.e. it decreases algebraically from 0 to \(-a\)].

In the third quadrant, the distance \(OM_3\) increases algebraically from \(-a\) to 0, and, in the fourth quadrant, the distance \(OM_4\) increases from 0 to \(a\).

In the first quadrant, the length \(M_1P_1\) increases from 0 to \(a\); in the second quadrant, \(M_2P_2\) decreases from \(a\) to 0; in the third quadrant, \(M_3P_3\) decreases algebraically from 0 to \(-a\); whilst in the fourth quadrant \(M_4P_4\) increases algebraically from \(-a\) to 0.

54. Sine. In the first quadrant, as the angle increases from 0 to 90°, the sine, i.e. \(\frac{M_1P_1}{a}\), increases from \(\frac{0}{a}\) to \(\frac{a}{a}\), i.e. from 0 to 1.

In the second quadrant, as the angle increases from 90° to 180°, the sine decreases from \(\frac{a}{a}\) to \(\frac{0}{a}\), i.e. from 1 to 0.

In the third quadrant, as the angle increases from 180° to 270°, the sine decreases from \(\frac{0}{a}\) to \(\frac{-a}{a}\), i.e. from 0 to \(-1\).
In the fourth quadrant, as the angle increases from 270° to 360°, the sine increases from $-\frac{a}{a}$ to $\frac{0}{a}$, i.e. from $-1$ to 0.

55. **Cosine.** In the first quadrant the cosine, which is equal to $\frac{OM}{a}$, decreases from $\frac{a}{a}$ to $\frac{0}{a}$, i.e. from 1 to 0.

In the second quadrant, it decreases from $\frac{0}{a}$ to $\frac{-a}{a}$, i.e. from 0 to $-1$.

In the third quadrant, it increases from $\frac{-a}{a}$ to $\frac{0}{a}$, i.e. from $-1$ to 0.

In the fourth quadrant, it increases from $\frac{0}{a}$ to $\frac{a}{a}$, i.e. from 0 to 1.

56. **Tangent.** In the first quadrant, $M_1P_1$ increases from 0 to $a$ and $OM_1$ decreases from $a$ to 0, so that $\frac{M_1P_1}{OM_1}$ continually increases (for its numerator continually increases and its denominator continually decreases).

When $OP_1$ coincides with $OA$, the tangent is 0; when the revolving line has turned through an angle which is slightly less than a right angle, so that $OP_1$ nearly coincides with $OB$, then $M_1P_1$ is very nearly equal to $a$ and $OM_1$ is very small. The ratio $\frac{M_1P_1}{OM_1}$ is therefore very large, and the nearer $OP_1$ gets to $OB$ the larger does the ratio become, so that, by taking the revolving line near enough to $OB$, we can make the tangent as large as we please. This is expressed by saying that, when the angle is equal to 90°, its tangent is infinite.
The symbol $\infty$ is used to denote an infinitely great quantity.

Hence in the first quadrant the tangent increases from 0 to $\infty$.

In the second quadrant, when the revolving line has described an angle $AOP_2$ slightly greater than a right angle, $M_2P_2$ is very nearly equal to $a$ and $OM_2$ is very small and negative, so that the corresponding tangent is very large and negative.

Also, as the revolving line turns from $OB$ to $OA'$, $M_2P_2$ decreases from $a$ to 0 and $OM_2$ is negative and decreases from 0 to $-a$, so that when the revolving line coincides with $OA'$ the tangent is zero.

Hence, in the second quadrant, the tangent increases from $-\infty$ to 0.

In the third quadrant, both $M_3P_3$ and $OM_3$ are negative, and hence their ratio is positive. Also, when the revolving line coincides with $OB'$, the tangent is infinite.

Hence, in the third quadrant, the tangent increases from 0 to $\infty$.

In the fourth quadrant, $M_4P_4$ is negative and $OM_4$ is positive, so that their ratio is negative. Also, as the revolving line passes through $OB'$ the tangent changes from $+\infty$ to $-\infty$ [just as in passing through $OB$].

Hence, in the fourth quadrant, the tangent increases from $-\infty$ to 0.

57. Cotangent. When the revolving line coincides with $OA$, $M_1P_1$ is very small and $OM_1$ is very nearly equal to $a$, so that the cotangent, i.e. the ratio $\frac{OM_1}{M_1P_1}$, is infinite to start with. Also, as the revolving line rotates
from $OA$ to $OB$, the quantity $M_1P_1$ increases from 0 to $a$ and $OM_1$ decreases from $a$ to 0.

Hence, in the first quadrant, the cotangent decreases from $\infty$ to 0.

In the second quadrant, $M_2P_2$ is positive and $OM_2$ negative, so that the cotangent decreases from 0 to $\frac{-a}{0}$, i.e. from 0 to $-\infty$.

In the third quadrant, it is positive and decreases from $\infty$ to 0 [for as the revolving line crosses $OA$ the cotangent changes from $-\infty$ to $\infty$].

In the fourth quadrant, it is negative and decreases from 0 to $-\infty$.

58. **Secant.** When the revolving line coincides with $OA$ the value of $OM_1$ is $a$, so that the value of the secant is then unity.

As the revolving line turns from $OA$ to $OB$, $OM_1$ decreases from $a$ to 0, and when the revolving line coincides with $OB$ the value of the secant is $\frac{a}{0}$, i.e. $\infty$.

Hence, in the first quadrant, the secant increases from 1 to $\infty$.

In the second quadrant, $OM_2$ is negative and decreases from 0 to $-a$. Hence, in this quadrant, the secant increases from $-\infty$ to $-1$ [for as the revolving line crosses $OB$ the quantity $OM_1$ changes sign and therefore the secant changes from $+\infty$ to $-\infty$].

In the third quadrant, $OM_3$ is always negative and increases from $-a$ to 0; therefore the secant decreases from $-1$ to $-\infty$. In the fourth quadrant, $OM_4$ is always positive and increases from 0 to $a$. Hence, in this quadrant, the secant decreases from $\infty$ to $+1$. 
59. Cosecant. The change in the cosecant may be traced in a similar manner to that in the secant.

In the first quadrant, it decreases from $\infty$ to $+1$.
In the second quadrant, it increases from $+1$ to $+\infty$.
In the third quadrant, it increases from $-\infty$ to $-1$.
In the fourth quadrant, it decreases from $-1$ to $-\infty$.

60. The foregoing results are collected in the annexed table.

<table>
<thead>
<tr>
<th>In the second quadrant, the</th>
<th>B</th>
<th>In the first quadrant, the</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine decreases from 1 to 0</td>
<td>B</td>
<td>sine increases from 0 to 1</td>
</tr>
<tr>
<td>cosine decreases from 0 to -1</td>
<td></td>
<td>cosine decreases from 1 to 0</td>
</tr>
<tr>
<td>tangent increases from $-\infty$ to 0</td>
<td></td>
<td>tangent increases from 0 to $\infty$</td>
</tr>
<tr>
<td>cotangent decreases from 0 to $-\infty$</td>
<td></td>
<td>cotangent decreases from $\infty$ to 0</td>
</tr>
<tr>
<td>secant increases from $-\infty$ to 1</td>
<td></td>
<td>secant increases from 1 to $\infty$</td>
</tr>
<tr>
<td>cosecant increases from 1 to $\infty$</td>
<td></td>
<td>cosecant decreases from $\infty$ to 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>In the third quadrant, the</th>
<th></th>
<th>In the fourth quadrant, the</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine decreases from 0 to -1</td>
<td>B</td>
<td>sine increases from -1 to 0</td>
</tr>
<tr>
<td>cosine increases from -1 to 0</td>
<td></td>
<td>cosine increases from 0 to 1</td>
</tr>
<tr>
<td>tangent increases from 0 to $\infty$</td>
<td></td>
<td>tangent increases from $-\infty$ to 0</td>
</tr>
<tr>
<td>cotangent decreases from $\infty$ to 0</td>
<td></td>
<td>cotangent decreases from 0 to $-\infty$</td>
</tr>
<tr>
<td>secant decreases from -1 to $-\infty$</td>
<td></td>
<td>secant decreases from $\infty$ to 1</td>
</tr>
<tr>
<td>cosecant increases from $-\infty$ to -1</td>
<td></td>
<td>cosecant decreases from -1 to $-\infty$</td>
</tr>
</tbody>
</table>

61. Periods of the trigonometrical functions.

As an angle increases from 0 to $2\pi$ radians, i.e. whilst the revolving line makes a complete revolution, its sine first increases from 0 to 1, then decreases from 1 to $-1$, and finally increases from $-1$ to 0, and thus the sine goes through all its changes, returning to its original value.
Similarly, as the angle increases from $2\pi$ radians to $4\pi$ radians, the sine goes through the same series of changes.

Also, the sines of any two angles which differ by four right angles, i.e. $2\pi$ radians, are the same.

This is expressed by saying that the period of the sine is $2\pi$.

Similarly, the cosine, secant, and cosecant go through all their changes as the angle increases by $2\pi$.

The tangent, however, goes through all its changes as the angle increases from 0 to $\pi$ radians, i.e. whilst the revolving line turns through two right angles. Similarly for the cotangent.

The period of the sine, cosine, secant and cosecant is therefore $2\pi$ radians; the period of the tangent and cotangent is $\pi$ radians.

Since the values of the trigonometrical functions repeat over and over again as the angle increases, they are called periodic functions.

*62. The variations in the values of the trigonometrical ratios may be graphically represented to the eye by means of curves constructed in the following manner.

Sine-Graph.
Let $OX$ and $OY$ be two straight lines at right angles
and let the magnitudes of angles be represented by lengths measured along $OX$.

Let $R_1, R_2, R_3, \ldots$ be points such that the distances $OR_1, R_1R_2, R_2R_3, \ldots$ are equal. If then the distance $OR_1$ represent a right angle, the distances $OR_2, OR_3, OR_4, \ldots$ must represent two, three, four, $\ldots$ right angles.

Also, if $P$ be any point on the line $OX$, then $OP$ represents an angle which bears the same ratio to a right angle that $OP$ bears to $OR_1$.

[For example, if $OP$ be equal to $\frac{1}{3} OR_1$, then $OP$ would represent one-third of a right angle; if $P$ bisected $R_3R_4$, then $OP$ would represent $3\frac{1}{2}$ right angles.]

Let also $OR_1$ be so chosen that one unit of length represents one radian; since $OR_2$ represents two right angles, i.e. $\pi$ radians, the length $OR_2$ must be $\pi$ units of length, i.e. about $3\frac{1}{4}$ units of length.

In a similar manner, negative angles are represented by distances $OR_1', OR_2', \ldots$ measured from $O$ in a negative direction.

At each point $P$ erect a perpendicular $PQ$ to represent the sine of the angle which is represented by $OP$; if the sine be positive, the perpendicular is to be drawn parallel to $OY$ in the positive direction; if the sine be negative, the line is to be drawn in the negative direction.

[For example, since $OR_1$ represents a right angle, the sine of which is 1, we erect a perpendicular $R_1B_1$ equal to one unit of length; since $OR_2$ represents an angle equal to two right angles, the sine of which is zero, we erect a perpendicular of length zero; since $OR_3$ represents three right angles, the sine of which is $-1$, we erect a perpendicular equal to $-1$, i.e. we draw $R_3B_3$ downward and equal to a unit of length; if $OP$ were equal to one-third of $OR_1$, it would represent $\frac{1}{3}$ of a right angle, i.e. $30^\circ$.]
the sine of which is \( \frac{1}{2} \), and so we should erect a perpendicular \( PQ \) equal to one-half the unit of length.]

The ends of all these lines, thus drawn, would be found to lie on a curve similar to the one drawn above.

It would be found that the curve consisted of portions, similar to \( OB_1R_2B_3R_4 \), placed side by side. This corresponds to the fact that each time the angle increases by \( 2\pi \), the sine repeats the same value.

*63. Cosine-Graph.

![Cosine-Graph Diagram](image)

The Cosine-Graph is obtained in the same manner as the Sine-Graph, except that in this case the perpendicular \( PQ \) represents the cosine of the angle represented by \( OP \).

The curve obtained is the same as that of Art. 62 if in that curve we move \( O \) to \( R_1 \) and let \( OY \) be drawn along \( R_1B_1 \).

*64. Tangent-Graph.

In this case, since the tangent of a right angle is infinite and since \( OR_1 \) represents a right angle, the perpendicular drawn at \( R_1 \) must be of infinite length and the dotted curve will only meet the line \( R_1L \) at an infinite distance.
Since the tangent of an angle slightly greater than a right angle is negative and almost infinitely great, the dotted curve immediately beyond $LR_1L'$ commences at an infinite distance on the negative side, i.e. below, $OX$.

The Tangent-Graph will clearly consist of an infinite number of similar but disconnected portions, all ranged parallel to one another. Such a curve is called a Discontinuous Curve. Both the Sine-Graph and the Cosine-Graph are, on the other hand, Continuous Curves.

*65. Cotangent-Graph. If the curve to represent the cotangent be drawn in a similar manner, it will be found to meet $OY$ at an infinite distance above $O$; it will pass through the point $R_1$ and touch the vertical line through $R_2$ at an infinite distance on the negative side of $OX$. Just beyond $R_2$ it will start at an infinite distance above $R_1$, and proceed as before.

The curve is therefore discontinuous and will consist of an infinite number of portions all ranged side by side.
66. Cosecant-Graph.

When the angle is zero, the sine is zero, and the cosecant is therefore infinite.

Hence the curve meets $OY$ at infinity.

When the angle is a right angle, the cosecant is unity, and hence $R_1B_1$ is equal to the unit of length.

When the angle is equal to two right angles its cosecant is infinity, so that the curve meets the perpendicular through $R_2$ at an infinite distance.

Again, as the angle increases from slightly less to slightly greater than two right angles, the cosecant changes from $+\infty$ to $-\infty$.

Hence just beyond $R_2$ the curve commences at an infinite distance on the negative side of, i.e. below, $OX$.

*67. Secant-Graph. If, similarly, the Secant-Graph be traced it will be found to be the same as the Cosecant-Graph would be if we moved $OY$ to $R_1B_1$.

[Some further examples of graphs will be found on pages 144, 145, 158 and 281.]
MISCELLANEOUS EXAMPLES. IX.

1. In a triangle one angle contains as many grades as another contains degrees, and the third contains as many centesimal seconds as there are sexagesimal seconds in the sum of the other two; find the number of radians in each angle.

2. Find the number of degrees, minutes, and seconds in the angle at the centre of a circle, whose radius is 5 feet, which is subtended by an arc of length 6 feet.

3. To turn radians into seconds, prove that we must multiply by 206265 nearly, and to turn seconds into radians the multiplier must be 0.0000048.

4. If \( \sin \theta = \frac{x^2 - y^2}{x^2 + y^2} \), find the values of \( \cos \theta \) and \( \cot \theta \).

5. If \( \sin \theta = \frac{m^2 + 2mn}{m^2 + 2mn + 2n^2} \), prove that \( \tan \theta = \frac{m^2 + 2mn}{2mn + 2n^2} \).

6. If \( \cos \theta - \sin \theta = \sqrt{2} \sin \theta \), prove that \( \cos \theta + \sin \theta = \sqrt{2} \cos \theta \).

7. Prove that \( \csc^6 \alpha - \cot^6 \alpha = 3 \csc^2 \alpha \cot^2 \alpha + 1 \).

8. Express \( 2 \sec^2 A - \sec^4 A - 2 \csc^2 A + \csc^4 A \) in terms of \( \tan A \).

9. Solve the equation \( 3 \csc^2 \theta = 2 \sec \theta \).

10. A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore?

11. Prove that the equation \( \sin \theta = x + \frac{1}{x} \) is impossible if \( x \) be real.

12. Shew that the equation \( \sec^2 \theta = \frac{4xy}{(x + y)^2} \) is only possible when \( x = y \).
CHAPTER V.

TRIGONOMETRICAL FUNCTIONS OF ANGLES OF ANY SIZE AND SIGN.

[On a first reading of the subject, the student is recommended to confine his attention to the first of the four figures given in Arts. 68, 69, 70, and 72.]

68. To find the trigonometrical ratios of an angle \((-\theta)\) in terms of those of \(\theta\), for all values of \(\theta\).
Let the revolving line, starting from $OA$, revolve through any angle $\theta$ and stop in the position $OP$.

Draw $PM$ perpendicular to $OA$ (or $OA$ produced) and produce it to $P'$, so that the lengths of $PM$ and $MP'$ are equal.

In the geometrical triangles $MOP$ and $MOP'$, we have the two sides $OM$ and $MP$ equal to the two $OM$ and $MP'$, and the included angles $OMP$ and $OMP'$ are right angles.

Hence (Euc. 1. 4), the magnitudes of the angles $MOP$ and $MOP'$ are the same, and $OP$ is equal to $OP'$.

In each of the four figures, the magnitudes of the angle $AOP$ (measured counter-clockwise) and of the angle $AOP'$ (measured clockwise) are the same.

Hence the angle $AOP'$ (measured clockwise) is denoted by $-\theta$.

Also $MP$ and $MP'$ are equal in magnitude but are opposite in sign. (Art. 49.) We have therefore

$$\sin (-\theta) = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\sin \theta,$$

$$\cos (-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos \theta,$$

$$\tan (-\theta) = \frac{MP'}{OM} = -\frac{MP}{OM} = -\tan \theta,$$

$$\cot (-\theta) = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot \theta,$$

$$\cosec (-\theta) = \frac{OP'}{MP'} = \frac{OP}{-MP} = -\cosec \theta,$$

and $$\sec (-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec \theta.$$
[In this article, and the following articles, the values of the last four
trigonometrical ratios may be found, without reference to the figure,
from the values of the first two ratios.

Thus
\[ \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta, \]
\[ \cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta, \]
\[ \csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta, \]
and
\[ \sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta. \]

Exs.
\[ \sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2}, \]
\[ \tan(-60^\circ) = -\tan 60^\circ = -\sqrt{3}, \]
and
\[ \cos(-45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}. \]

69. To find the trigonometrical ratios of the angle
\((90^\circ - \theta)\) in terms of those of \(\theta\), for all values of \(\theta\).

The relations have already been discussed in Art. 39,
for values of \(\theta\) less than a right angle.

Let the revolving line, starting from \(OA\), trace out
any angle \(AOP\) denoted by \(\theta\).

To obtain the angle \(90^\circ - \theta\), let the revolving line
rotate to \(B\) and then rotate from \(B\) in the opposite
direction through the angle \(\theta\), and let the position of the
revolving line be then \(OP'\).

The angle \(AOP'\) is then \(90^\circ - \theta\).

Take \(OP'\) equal to \(OP\), and draw \(P'M'\) and \(PM\) per-
pendicular to \(OA\), produced if necessary. Also draw \(P'N'\)
perpendicular to \(OB\), produced if necessary.
In each figure, the angles $AOP$ and $BOP'$ are numerically equal, by construction.

\[ \angle MOP = \angle N'OP' = \angle OP'M', \]

since $ON'$ and $MP'$ are parallel.

Hence the triangles $MOP$ and $M'P'O$ are equal in all respects, and therefore $OM = M'P'$ numerically, and $OM' = MP$ numerically.

Also, in each figure, $OM$ and $M'P'$ are of the same sign, and so also are $MP$ and $OM'$,

\[ i.e. \ OM = +M'P', \ \text{and} \ OM' = +MP. \]
Hence

\[ \sin (90^\circ - \theta) = \sin AOP' = \frac{MP'}{OP'} = \frac{OM}{OP} = \cos \theta, \]

\[ \cos (90^\circ - \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin \theta, \]

\[ \tan (90^\circ - \theta) = \tan AOP' = \frac{MP'}{OM'} = \frac{OM}{MP} = \cot \theta, \]

\[ \cot (90^\circ - \theta) = \cot AOP' = \frac{OM'}{MP'} = \frac{MP}{OM} = \tan \theta, \]

\[ \sec (90^\circ - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{MP} = \cosec \theta, \]

and \[ \cosec (90^\circ - \theta) = \cosec AOP' = \frac{OP'}{MP'} = \frac{OP}{OM} = \sec \theta. \]

70. To find the trigonometrical ratios of the angle \((90^\circ + \theta)\) in terms of those of \(\theta\), for all values of \(\theta\).
Let the revolving line, starting from \( OA \), trace out any angle \( \theta \) and let \( OP \) be the position of the revolving line then, so that the angle \( AOP \) is \( \theta \).

Let the revolving line turn through a right angle from \( OP \) in the positive direction to the position \( OP' \), so that the angle \( AOP' \) is \( (90^\circ + \theta) \).

Take \( OP' \) equal to \( OP \) and draw \( PM \) and \( P'M' \) perpendicular to \( AO \), produced if necessary. In each figure, since \( POP' \) is a right angle, the sum of the angles \( MOP \) and \( P'OM' \) is always a right angle.

Hence \( \angle MOP = 90^\circ - \angle P'OM' = \angle OP'M' \).

The two triangles \( MOP \) and \( M'P'O \) are therefore equal in all respects.

Hence \( OM \) and \( MP' \) are numerically equal, as also \( MP \) and \( OM' \) are numerically equal.

In each figure, \( OM \) and \( MP' \) have the same sign, whilst \( MP \) and \( OM' \) have the opposite sign, so that \( MP' = + OM \), and \( OM' = - MP \).

We therefore have

\[
\sin (90^\circ + \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OF} = \cos \theta,
\]

\[
\cos (90^\circ + \theta) = \cos AOP' = \frac{OM}{OP'} = \frac{-MP}{OP} = - \sin \theta,
\]

\[
\tan (90^\circ + \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{MP} = - \cot \theta,
\]

\[
\cot (90^\circ + \theta) = \cot AOP' = \frac{OM}{M'P'} = \frac{-MP}{OM} = - \tan \theta,
\]

\[
\sec (90^\circ + \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{MP} = - \csc \theta,
\]

and \( \csc (90^\circ + \theta) = \csc AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec \theta \).
\[ \text{Exs.} \quad \sin 150^\circ = \sin (90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}, \]
\[ \cos 135^\circ = \cos (90^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}, \]
and \[ \tan 120^\circ = \tan (90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}. \]

71. **Supplementary Angles.**

Two angles are said to be supplementary when their sum is equal to two right angles, \textit{i.e.} the supplement of any angle \( \theta \) is \( 180^\circ - \theta \).

\[ \text{Exs.} \quad \text{The supplement of } 30^\circ = 180^\circ - 30^\circ = 150^\circ. \]
\[ \text{The supplement of } 120^\circ = 180^\circ - 120^\circ = 60^\circ. \]
\[ \text{The supplement of } 275^\circ = 180^\circ - 275^\circ = -95^\circ. \]
\[ \text{The supplement of } -126^\circ = 180^\circ - (-126^\circ) = 306^\circ. \]

72. **To find the values of the trigonometrical ratios of the angle** \( (180^\circ - \theta) \) **in terms of those of the angle** \( \theta \), **for all values of** \( \theta \).

Let the revolving line start from \( OA \) and describe any angle \( AOP (= \theta) \).
ANGLES OF ANY SIZE AND SIGN. 71

To obtain the angle $180^\circ - \theta$, let the revolving line start from $OA$ and, after revolving through two right angles (i.e. into the position $OA'$), then revolve back through an angle $\theta$ into the position $OP'$, so that the angle $A'OP'$ is equal in magnitude but opposite in sign to the angle $AOP$.

The angle $AOP'$ is then $180^\circ - \theta$.

Take $OP'$ equal to $OP$, and draw $PM'$ and $PM$ perpendicular to $AOA'$.

The angles $MOP$ and $M'OP'$ are equal, and hence the triangles $MOP$ and $M'OP'$ are equal in all respects.

Hence $OM$ and $OM'$ are equal in magnitude, and so also are $MP$ and $M'P'$.

In each figure, $OM$ and $OM'$ are drawn in opposite directions, whilst $MP$ and $M'P'$ are drawn in the same direction, so that

$$OM' = -OM, \text{ and } M'P' = +MP.$$  

Hence we have

$$\sin (180^\circ - \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin \theta,$$

$$\cos (180^\circ - \theta) = \cos AOP' = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos \theta,$$

$$\tan (180^\circ - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan \theta,$$

$$\cot (180^\circ - \theta) = \cot AOP' = \frac{OM'}{M'P'} = -\frac{OM}{MP} = -\cot \theta,$$

$$\sec (180^\circ - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec \theta,$$

and $$\cosec (180^\circ - \theta) = \cosec AOP' = \frac{OP'}{M'P'} = \frac{OP}{MP} = \cosec \theta.$$
\[ \text{Exs.} \quad \sin 120^\circ = \sin (180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}, \]
\[ \cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}, \]
and \[ \tan 150^\circ = \tan (180^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}. \]

73. To find the trigonometrical ratios of \((180^\circ + \theta)\) in terms of those of \(\theta\), for all values of \(\theta\).

The required relations may be obtained geometrically, as in the previous articles. The figures for this proposition are easily obtained and are left as an example for the student.

They may also be deduced from the results of Art. 70, which have been proved true for all angles. For putting \(90^\circ + \theta = B\), we have

\[ \sin (180^\circ + \theta) = \sin (90^\circ + B) = \cos B \quad \text{(Art. 70)} \]
\[ = \cos (90^\circ + \theta) = -\sin \theta, \quad \text{(Art. 70)} \]
and \[ \cos (180^\circ + \theta) = \cos (90^\circ + B) = -\sin B \quad \text{(Art. 70)} \]
\[ = -\sin (90^\circ + \theta) = -\cos \theta. \quad \text{(Art. 70).} \]

So \[ \tan (180^\circ + \theta) = \tan (90^\circ + B) = -\cot B \]
\[ = -\cot (90^\circ + \theta) = \tan \theta, \]
and similarly \[ \cot (180^\circ + \theta) = \cot \theta, \]
\[ \sec (180^\circ + \theta) = -\sec \theta, \]
and \[ \cosec (180^\circ + \theta) = -\cosec \theta. \]

74. To find the trigonometrical ratios of an angle \((360^\circ + \theta)\) in terms of those of \(\theta\), for all values of \(\theta\).
ANGLES OF ANY SIZE AND SIGN.

In whatever position the revolving line may be when it has described any angle $\theta$, it will be in exactly the same position when it has made one more complete revolution in the positive direction, i.e. when it has described an angle $360^\circ + \theta$.

Hence the trigonometrical ratios for an angle $360^\circ + \theta$ are the same as those for $\theta$.

It follows that the addition or subtraction of $360^\circ$, or any multiple of $360^\circ$, to or from any angle does not alter its trigonometrical ratios.

75. From the theorems of this chapter it follows that the trigonometrical ratios of any angle whatever can be reduced to the determination of the trigonometrical ratios of an angle which lies between $0^\circ$ and $45^\circ$.

For example,

\[
\sin 1765^\circ = \sin [4 \times 360^\circ + 325^\circ] = \sin 325^\circ \quad \text{(Art. 74)}
\]

\[
= \sin (180^\circ + 145^\circ) = - \sin 145^\circ \quad \text{(Art. 73)}
\]

\[
= - \sin (180^\circ - 35^\circ) = - \sin 35^\circ \quad \text{(Art. 72)};
\]

\[
\tan 1190^\circ = \tan (3 \times 360^\circ + 110^\circ) = \tan 110^\circ \quad \text{(Art. 74)}
\]

\[
= \tan (90^\circ + 20^\circ) = - \cot 20^\circ \quad \text{(Art. 70)};
\]

and \[
\cosec (- 1465^\circ) = - \cosec 1465^\circ \quad \text{(Art. 68)}
\]

\[
= - \cosec (4 \times 360^\circ + 25^\circ) = - \cosec 25^\circ \quad \text{(Art. 74)}.
\]

Similarly any other such large angles may be treated. First, multiples of $360^\circ$ should be subtracted until the angle lies between $0^\circ$ and $360^\circ$; if it be then greater than $180^\circ$, it should be reduced by $180^\circ$; if then greater than $90^\circ$, the formulae of Art. 70 should be used, and finally, if necessary, the formulae of Art. 69 applied.
76. The table of Art. 40 may now be extended to some important angles greater than a right angle.

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>√3/2</td>
<td>1</td>
<td>√3/2</td>
<td>1/2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Cosine</td>
<td>1</td>
<td>√3/2</td>
<td>1/2</td>
<td>1</td>
<td>√3/2</td>
<td>1/2</td>
<td>√3/2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Tangent</td>
<td>0</td>
<td>1/√3</td>
<td>1</td>
<td>√3</td>
<td>∞</td>
<td>-√3</td>
<td>-1</td>
<td>-1/√3</td>
<td>0</td>
</tr>
<tr>
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<td>1/2</td>
<td>1/√3</td>
<td>0</td>
<td>-1/√3</td>
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<td>-√3</td>
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<tr>
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<td>√3/3</td>
<td>2</td>
<td>1</td>
<td>2/√3</td>
<td>√2</td>
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<tr>
<td>Secant</td>
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<td>-2</td>
<td>-√2</td>
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<td>-1</td>
</tr>
</tbody>
</table>

**EXAMPLES. X.**

Prove that

1. \( \sin 420° \cos 390° + \cos (-300°) \sin (-330°) = 1. \)
2. \( \cos 570° \sin 510° - \sin 330° \cos 390° = 0. \)

and 3. \( \tan 225° \cot 405° + \tan 765° \cot 675° = 0. \)

What are the values of \( \cos A - \sin A \) and \( \tan A + \cot A \) when \( A \) has the values

4. \( \frac{\pi}{8}, \) 5. \( \frac{2\pi}{3}, \) 6. \( \frac{5\pi}{4}, \) 7. \( \frac{7\pi}{4} \) and 8. \( \frac{11\pi}{3} \)?
What values between $0^\circ$ and $360^\circ$ may $A$ have when

9. $\sin A = \frac{1}{\sqrt{2}}$,  
10. $\cos A = -\frac{1}{2}$,  
11. $\tan A = -1$,

12. $\cot A = -\sqrt{3}$,  
13. $\sec A = -\frac{2}{\sqrt{3}}$ and  
14. $\csc A = -2$?

Express in terms of the ratios of a positive angle, which is less than $45^\circ$, the quantities

15. $\sin (-65^\circ)$.  
16. $\cos (-84^\circ)$.  
17. $\tan 137^\circ$.

18. $\sin 168^\circ$.  
19. $\cos 287^\circ$.  
20. $\tan (-246^\circ)$.

21. $\sin 843^\circ$.  
22. $\cos (-928^\circ)$.  
23. $\tan 1145^\circ$.

24. $\cos 1410^\circ$.  
25. $\cot (-1054^\circ)$.  
26. $\sec 1327^\circ$ and

27. $\csc (-756^\circ)$.

What sign has $\sin A + \cos A$ for the following values of $A$?

28. $140^\circ$.  
29. $278^\circ$.  
30. $-356^\circ$ and  
31. $-1125^\circ$.

What sign has $\sin A - \cos A$ for the following values of $A$?

32. $215^\circ$.  
33. $825^\circ$.  
34. $-634^\circ$ and  
35. $-457^\circ$.

36. Find the sines and cosines of all angles in the first four quadrants whose tangents are equal to $\cos 135^\circ$.

Prove that

37. $\sin (270^\circ + A) = -\cos A$, and $\tan (270^\circ + A) = -\cot A$.

38. $\cos (270^\circ - A) = -\sin A$, and $\cot (270^\circ - A) = \tan A$.

39. $\cos A + \sin (270^\circ + A) - \sin (270^\circ - A) + \cos (180^\circ + A) = 0$.

40. $\sec (270^\circ - A) \sec (90^\circ - A) - \tan (270^\circ - A) \tan (90^\circ + A) + 1 = 0$.

41. $\cot A + \tan (180^\circ + A) + \tan (90^\circ + A) + \tan (360^\circ - A) = 0$. 
CHAPTER VI.

GENERAL EXPRESSIONS FOR ALL ANGLES HAVING A GIVEN TRIGONOMETRICAL RATIO.

77. To construct the least positive angle whose sine is equal to \( a \), where \( a \) is a proper fraction.

Let \( OA \) be the initial line, and let \( OB \) be drawn in the positive direction perpendicular to \( OA \).

Measure off along \( OB \) a distance \( ON \) which is equal to \( a \) units of length. [If \( a \) be negative the point \( N \) will lie in \( BO \) produced.]

Through \( N \) draw \( NP \) parallel to \( OA \). With centre \( O \), and radius equal to the unit of length, describe a circle and let it meet \( NP \) in \( P \).

Then \( AOP \) will be the required angle.

Draw \( PM \) perpendicular to \( OA \), so that

\[
\sin AOP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{a}{1} = a.
\]

The sine of \( AOP \) is therefore equal to the given quantity, and hence \( AOP \) is the angle required.
78. **To construct the least positive angle whose cosine is equal to \( b \), where \( b \) is a proper fraction.**

Along the initial line measure off a distance \( OM \) equal to \( b \) and draw \( MP \) perpendicular to \( OA \). [If \( b \) be negative, \( M \) will lie on the other side of \( O \) in the line \( AO \) produced.]

With centre \( O \), and radius equal to unity, describe a circle and let it meet \( MP \) in \( P \).

Then \( AOP \) is the angle required. For

\[
\cos AOP = \frac{OM}{OP} = \frac{b}{1} = b.
\]

79. **To construct the least positive angle whose tangent is equal to \( c \).**

Along the initial line measure off \( OM \) equal to unity, and erect a perpendicular \( MP \). Measure off \( MP \) equal to \( c \).

Then

\[
\tan AOP = \frac{MP}{OM} = c,
\]

so that \( AOP \) is the required angle.

80. It is clear from the definition given in Art. 50, that, when an angle is given, so also is its sine. The converse statement is not correct; there is more than one angle having a given sine; for example, the angles 30°, 150°, 390°, −210°,... all have their sine equal to \( \frac{1}{2} \).

Hence, when the sine of an angle is given, we do not definitely know the angle; all we know is that the angle is one out of a large number of angles.
Similar statements are true if the cosine, tangent, or any other trigonometrical function of the angle be given.

Hence, simply to give one of the trigonometrical functions of an angle does not determine it without ambiguity.

81. Suppose we know that the revolving line $OP$ coincides with the initial line $OA$. All we know is that the revolving line has made 0, or 1, or 2, or 3, ... complete revolutions, either positive or negative.

But when the revolving line has made one complete revolution, the angle it has described is (Art. 17) equal to $2\pi$ radians.

Hence, when the revolving line $OP$ coincides with the initial line $OA$, the angle that it has described is 0, or 1, or 2, or 3 ... times $2\pi$ radians, in either the positive or negative directions, \(i.e.\) either 0, or \(\pm 2\pi\), or \(\pm 4\pi\), or \(\pm 6\pi\) ... radians.

This is expressed by saying that when the revolving line coincides with the initial line the angle it has described is $2n\pi$, where $n$ is some positive or negative integer.

82. Theorem. To find a general expression to include all angles which have the same sine.

Let $AOP$ be any angle having the given sine, and let it be denoted by $\alpha$.

Draw $PM$ perpendicular to $OA$ and produce $MO$ to $M'$, making $OM'$ equal to $MO$, and draw $M'P'$ parallel and equal to $MP$.

As in Art. 72, the angle $AOP'$ is equal to $\pi - \alpha$. 
ANGLES HAVING THE SAME SINE.

When the revolving line is in either of the positions \( OP \) or \( OP' \), and in no other position, the sine of the angle traced out is equal to the given sine.

When the revolving line is in the position \( OP \), it has made a whole number of complete revolutions and then described an angle \( \alpha \), i.e., by the last article, it has described an angle equal to

\[
2r\pi + \alpha \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

where \( r \) is zero or some positive or negative integer.

When the revolving line is in the position \( OP' \), it has, similarly, described an angle \( 2r\pi + AOP', \) i.e. an angle \( 2r\pi + \pi - \alpha, \) i.e. \( (2r + 1)\pi - \alpha \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)\)

where \( r \) is zero or some positive or negative integer.

All these angles will be found to be included in the expression

\[
n\pi + (-1)^n \alpha \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3),
\]

where \( n \) is zero or a positive or negative integer.

For, when \( n = 2r \), since \((-1)^{2r} = +1\), the expression (3) gives \( 2r\pi + \alpha \), which is the same as the expression (1).

Also, when \( n = 2r + 1 \), since \((-1)^{2r+1} = -1\), the expression (3) gives \( (2r + 1)\pi - \alpha \), which is the same as the expression (2).

Cor. Since all angles which have the same sine have also the same cosecant, the expression (3) includes all angles which have the same cosecant as \( \alpha \).

83. Theorem. To find a general expression to include all angles which have the same cosine.

Let \( AOP \) be any angle having the given cosine, and let it be denoted by \( \alpha \).
Draw $PM$ perpendicular to $OA$ and produce it to $P'$, making $MP'$ equal to $PM$.

When the revolving line is in the position $OP$ or $OP'$, and in no other position, then, as in Art. 78, the cosine of the angle traced out is equal to the given cosine.

When the revolving line is in the position $OP$, it has made a whole number of complete revolutions and then described an angle $\alpha$, i.e. it has described an angle $2n\pi + \alpha$, where $n$ is zero or some positive or negative integer.

When the revolving line is in the position $OP'$, it has made a whole number of complete revolutions and then described an angle $-\alpha$, i.e. it has described an angle $2n\pi - \alpha$.

All these angles are included in the expression

$$2n\pi \pm \alpha$$

where $n$ is zero or some positive or negative integer.

**Cor.** The expression (1) includes all angles having the same secant as $\alpha$.

**84. Theorem.** To find a general expression for all angles which have the same tangent.

Let $AOP$ be any angle having the given tangent, and let it be denoted by $\alpha$.

Produce $PO$ to $P'$, making $OP'$ equal to $OP$, and draw $P'M'$ perpendicular to $OM'$.

As in Art. 73, the angles $AOP$ and $AOP'$ have the same tangent; also the angle $AOP' = \pi + \alpha$.

When the revolving line is in
the position $OP$, it has described a whole number of complete revolutions and then turned through an angle $\alpha$, i.e. it has described an angle

$$2r\pi + \alpha\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $r$ is zero or some positive or negative integer.

When the revolving line is in the position $OP'$, it has similarly described an angle $2r\pi + (\pi + \alpha)$,

\[ (2r + 1)\pi + \alpha\ldots\ldots\ldots\ldots\ldots\ldots(2). \]

All these angles are included in the expression

$$n\pi + \alpha\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $n$ is zero or some positive or negative integer.

For, when $n$ is even, ($= 2r$ say), the expression (3) gives the same angles as the expression (1).

Also, when $n$ is odd, ($= 2r + 1$ say), it gives the same angles as the expression (2).

**Cor.** The expression (3) includes all angles which have the same cotangent as $\alpha$.

85. In Arts. 82, 83, and 84 the angle $\alpha$ is any angle satisfying the given condition. In practical examples it is, in general, desirable to take $\alpha$ as the smallest positive angle which is suitable.

**Ex. 1.** Write down the general expression for all angles,

1. whose sine is equal to $\frac{\sqrt{3}}{2}$,
2. whose cosine is equal to $-\frac{1}{2}$,
3. whose tangent is equal to $\frac{1}{\sqrt{3}}$.

4. The smallest angle, whose sine is $\frac{\sqrt{3}}{2}$, is 60°, i.e. $\frac{\pi}{3}$.

L. T.
Hence, by Art. 82, the general expression for all the angles which have this sine is

\[ n\pi + (-1)^n \frac{\pi}{3}. \]

(2) The smallest positive angle, whose cosine is \(-\frac{1}{2}\),

is 120°, i.e. \(\frac{2\pi}{3}\).

Hence, by Art. 83, the general expression for all the angles which have this cosine is

\[ 2n\pi \pm \frac{2\pi}{3}. \]

(3) The smallest positive angle, whose tangent is \(\frac{1}{\sqrt{3}}\),

is 30°, i.e. \(\frac{\pi}{6}\).

Hence, by Art. 84, the general expression for all the angles which have this tangent is

\[ n\pi + \frac{\pi}{6}. \]

**Ex. 2.** What is the most general value of \(\theta\) satisfying the equation

\[ \sin^2 \theta = \frac{1}{4} ? \]

Here we have \(\sin \theta = \pm \frac{1}{2}\).

Taking the upper sign,

\[ \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6}. \]

\[ \therefore \theta = n\pi + (-1)^n \frac{\pi}{6}. \]

Taking the lower sign,

\[ \sin \theta = -\frac{1}{2} = \sin \left(-\frac{\pi}{6}\right). \]

\[ \therefore \theta = n\pi + (-1)^n \left(-\frac{\pi}{6}\right). \]

Putting both solutions together, we have

\[ \theta = n\pi \pm \left(-1\right)^n \frac{\pi}{6}. \]

or, what is the same expression,

\[ \theta = n\pi \pm \frac{\pi}{6}. \]
**EXAMPLES.**

**Ex. 3.** What is the most general value of $\theta$ which satisfies both of the equations $\sin \theta = -\frac{1}{2}$ and $\tan \theta = \frac{1}{\sqrt{3}}$?

Considering only angles between $0^\circ$ and $360^\circ$, the only values of $\theta$, when $\sin \theta = -\frac{1}{2}$, are $210^\circ$ and $330^\circ$. Similarly, the only values of $\theta$, when $\tan \theta = \frac{1}{\sqrt{3}}$, are $30^\circ$ and $210^\circ$.

The only value of $\theta$, between $0^\circ$ and $360^\circ$, satisfying both conditions is therefore $210^\circ$, i.e. $\frac{7\pi}{6}$.

The most general value is hence obtained by adding any multiple of four right angles to this angle, and hence is $2n\pi + \frac{7\pi}{6}$, where $n$ is any positive or negative integer.

**EXAMPLES. XI.**

What are the most general values of $\theta$ which satisfy the equations.

1. $\sin \theta = \frac{1}{2}$.
2. $\sin \theta = -\frac{\sqrt{3}}{2}$.
3. $\sin \theta = \frac{1}{\sqrt{2}}$.
4. $\cos \theta = \frac{1}{2}$.
5. $\cos \theta = \frac{\sqrt{3}}{2}$.
6. $\cos \theta = -\frac{1}{\sqrt{2}}$.
7. $\tan \theta = \sqrt{3}$.
8. $\tan \theta = -1$.
9. $\cot \theta = 1$.
10. $\sec \theta = 2$.
11. $\cosec \theta = \frac{2}{\sqrt{3}}$.
12. $\sin^2 \theta = 1$.
13. $\cos^2 \theta = \frac{1}{4}$.
14. $\tan^2 \theta = \frac{1}{3}$.
15. $4\sin^2 \theta = 8$.
16. $2\cot^2 \theta = \cosec^2 \theta$.
17. $\sec^2 \theta = \frac{4}{3}$?
18. What is the most general value of $\theta$ that satisfies both of the equations $\cos \theta = -\frac{1}{\sqrt{2}}$ and $\tan \theta = 1$?
19. What is the most general value of $\theta$ that satisfies both of the equations $\cot \theta = -\sqrt{3}$ and $\cosec \theta = -2$?

6—2
20. If \( \cos (A - B) = \frac{1}{2} \), and \( \sin (A + B) = \frac{1}{2} \), find the smallest positive values of \( A \) and \( B \) and also their most general values.

21. If \( \tan (A - B) = 1 \), and \( \sec (A + B) = \frac{2}{\sqrt{3}} \), find the smallest positive values of \( A \) and \( B \) and also their most general values.

22. Find the angles between \( 0^\circ \) and \( 360^\circ \) which have respectively (1) their sines equal to \( \frac{\sqrt{3}}{2} \), (2) their cosines equal to \( -\frac{1}{2} \), and (3) their tangents equal to \( \frac{1}{\sqrt{3}} \).

23. Taking into consideration only angles between \( 0^\circ \) and \( 180^\circ \), how many values of \( x \) are there if (1) \( \sin x = \frac{5}{7} \), (2) \( \cos x = \frac{1}{5} \), (3) \( \cos x = -\frac{4}{5} \), (4) \( \tan x = \frac{2}{3} \), and (5) \( \cot x = -7 \) ?

24. Given the angle \( x \) construct the angle \( y \) if (1) \( \sin y = 2 \sin x \), (2) \( \tan y = 3 \tan x \), (3) \( \cos y = \frac{1}{2} \cos x \), and (4) \( \sec y = \sec x \).

25. Shew that the same angles are indicated by the two following formulae: (1) \( (2n - 1) \frac{\pi}{2} + (-1)^n \frac{\pi}{3} \), and (2) \( 2n\pi \pm \frac{\pi}{6} \), \( n \) being any integer.

26. Prove that the two formulae

\[ (1) \left( 2n + \frac{1}{2} \right) \pi \pm \alpha \quad \text{and} \quad (2) \ n\pi + (-1)^n \left( \frac{\pi}{2} - \alpha \right) \]

denote the same angles, \( n \) being any integer.

Illustrate by a figure.

27. If \( \theta - \alpha = n\pi + (-1)^n \beta \), prove that \( \theta = 2m\pi + \alpha + \beta \) or else that \( \theta = (2m + 1) \pi + \alpha - \beta \), where \( m \) and \( n \) are any integers.

28. If \( \cos p\theta + \cos q\theta = 0 \), prove that the different values of \( \theta \) form two arithmetical progressions in which the common differences are \( \frac{2\pi}{p + q} \) and \( \frac{2\pi}{p - q} \) respectively.

29. Construct the angle whose sine is \( \frac{8}{2 + \sqrt{5}} \).
86. An equation involving the trigonometrical ratios of an unknown angle is called a trigonometrical equation. The equation is not completely solved unless we obtain an expression for all the angles which satisfy it.

Some elementary types of equations are solved in the following article.

87. **Ex. 1.** Solve the equation \(2 \sin^2 x + \sqrt{3} \cos x + 1 = 0\).

The equation may be written

\[
2 - 2 \cos^2 x + \sqrt{3} \cos x + 1 = 0,
\]

i.e.

\[
2 \cos^2 x - \sqrt{3} \cos x - 3 = 0,
\]

i.e.

\[
(\cos x - \sqrt{3})(2 \cos x + \sqrt{3}) = 0.
\]

The equation is therefore satisfied by \(\cos x = \sqrt{3}\), or \(\cos x = -\frac{\sqrt{3}}{2}\).

Since the cosine of an angle cannot be numerically greater than unity, the first factor gives no solution.

The smallest positive angle, whose cosine is \(-\frac{\sqrt{3}}{2}\), is \(150^\circ\), i.e. \(\frac{5\pi}{6}\).

Hence the most general value of the angle, whose cosine is \(-\frac{\sqrt{3}}{2}\), is \(2n\pi \pm \frac{5\pi}{6}\). (Art. 83.)

This is the general solution of the given equation.

**Ex. 2.** Solve the equation \(\tan 5\theta = \cot 2\theta\).

The equation may be written

\[
\tan 5\theta = \tan \left(\frac{\pi}{2} - 2\theta\right).
\]

Now the most general value of the angle, that has the same tangent as \(\frac{\pi}{2} - 2\theta\), is, by Art. 84, \(n\pi + \frac{\pi}{2} - 2\theta\),

where \(n\) is any positive or negative integer.

The most general solution of the equation is therefore

\[
5\theta = n\pi + \frac{\pi}{2} - 2\theta.
\]

\[
\therefore \theta = \frac{1}{7} \left( n\pi + \frac{\pi}{2} \right),
\]

where \(n\) is any integer.
EXAMPLES. XII.

Solve the equations

1. \( \cos^2 \theta - \sin \theta - \frac{1}{4} = 0. \)
2. \( 2 \sin^2 \theta + 3 \cos \theta = 0. \)
3. \( 2\sqrt{3} \cos^2 \theta = \sin \theta. \)
4. \( \cos \theta + \cos^2 \theta = 1. \)
5. \( 4 \cos \theta - 3 \sec \theta = 2 \tan \theta. \)
6. \( \sin^2 \theta - 2 \cos \theta + \frac{1}{4} = 0. \)
7. \( \tan^2 \theta - (1 + \sqrt{3}) \tan \theta + \sqrt{3} = 0. \)
8. \( \cot^2 \theta + \left( \sqrt{3} + \frac{1}{\sqrt{3}} \right) \cot \theta + 1 = 0. \)
9. \( \cot \theta = \frac{a b \tan \theta}{a - b}. \)
10. \( \tan^2 \theta + \cot^2 \theta = 2. \)
11. \( \sec \theta - 1 = (\sqrt{2} - 1) \tan \theta. \)
12. \( 3 (\sec^2 \theta + \tan^2 \theta) = 5. \)
13. \( \cot \theta + \tan \theta = 2 \cosec \theta. \)
14. \( 4 \cos^2 \theta + \sqrt{3} = 2 (\sqrt{3} + 1) \cos \theta. \)
15. \( 3 \sin^2 \theta - 2 \sin \theta = 1. \)
16. \( \sin 5\theta = \frac{1}{\sqrt{2}}. \)
17. \( \sin 9\theta = \sin \theta. \)
18. \( \sin 3\theta = \sin 2\theta. \)
19. \( \cos m\theta = \cos n\theta. \)
20. \( \sin 2\theta = \cos 3\theta. \)
21. \( \cos 5\theta = \cos 4\theta. \)
22. \( \cos m\theta = \sin n\theta. \)
23. \( \cot \theta = \tan 8\theta. \)
24. \( \cot \theta = \tan n\theta. \)
25. \( \tan 2\theta = \tan \frac{2}{\theta}. \)
26. \( \tan 2\theta \tan \theta = 1. \)
27. \( \tan^3 3\theta = \cot^3 a. \)
28. \( \tan 3\theta = \cot \theta. \)
29. \( \tan^2 3\theta = \tan^2 a. \)
30. \( 3 \tan^2 \theta = 1. \)
31. \( \tan mx + \cot nx = 0. \)
32. \( \tan (\pi \cot \theta) = \cot (\pi \tan \theta). \)
33. \( \sin (\theta - \phi) = \frac{1}{2}, \) and \( \cos (\theta + \phi) = \frac{1}{2}. \)
34. \( \cos (2x + 3y) = \frac{1}{2}, \) and \( \cos (3x + 2y) = \frac{\sqrt{3}}{2}. \)
35. Find all the angles between 0° and 90° which satisfy the equation \( \sec^2 \theta \cosec^3 \theta + 2 \cosec^2 \theta = 8. \)
36. If \( \tan^2 \theta = \frac{5}{4}, \) find versin \( \theta \) and explain the double result.
37. If the coversin of an angle be \( \frac{1}{5}, \) find its cosine and cotangent.
CHAPTER VII.

TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

88. Theorem. To prove that
\[
\sin(A + B) = \sin A \cos B + \cos A \sin B, \quad \text{and} \quad \cos(A + B) = \cos A \cos B - \sin A \sin B.
\]

Let the revolving line start from \(OA\) and trace out the angle \(AOB (= A)\), and then trace out the further angle \(BOC (= B)\).

In the final position of the revolving line take any point \(P\), and draw \(PM\) and \(PN\) perpendicular to \(OA\) and \(OB\) respectively; through \(N\) draw \(NR\) parallel to \(AO\) to meet \(MP\) in \(R\), and draw \(NQ\) perpendicular to \(OA\).

The angle
\[
RPN = 90^\circ - \angle PNR = \angle RNO = \angle NOQ = A.
\]
Hence \( \sin (A + B) = \frac{MP}{OP} = \frac{MR + RP}{OP} \)
\[= \frac{QN}{OP} + \frac{RP}{OP} = \frac{QN}{ON} + \frac{NP}{NP} \frac{NP}{OP} \]
\[= \sin A \cos B + \cos RPN \sin B. \]
\[\therefore \sin (A + B) = \sin A \cos B + \cos A \sin B. \]

Again \( \cos (A + B) = \frac{OM}{OP} = \frac{OQ - MQ}{OP} \)
\[= \frac{OQ}{OP} \frac{RN}{OP} = \frac{OQ}{ON} \frac{NP}{NP} \frac{NP}{OP} \]
\[= \cos A \cos B - \sin RPN \sin B. \]
\[\therefore \cos (A + B) = \cos A \cos B - \sin A \sin B. \]

89. The figures in the last article have been drawn only for the case in which \( A \) and \( B \) are acute angles.

The same proof will be found to apply to angles of any size, due attention being paid to the signs of the quantities involved.

The results may however be shewn to be true of all angles, without drawing any more figures, as follows.

Let \( A \) and \( B \) be acute angles, so that, by Art. 88, we know that the theorem is true for \( A \) and \( B \).

Let \( A_1 = 90^\circ + A \), so that, by Art. 70, we have
\[ \sin A_1 = \cos A, \quad \text{and} \quad \cos A_1 = - \sin A. \]

Then \( \sin (A_1 + B) = \sin (90^\circ + (A + B)) = \cos (A + B), \) by Art. 70,
\[= \cos A \cos B - \sin A \sin B = \sin A_1 \cos B + \cos A_1 \sin B. \]

Also \( \cos (A_1 + B) = \cos (90^\circ + (A + B)) = - \sin (A + B) \)
\[= - \sin A \cos B - \cos A \sin B = \cos A_1 \cos B - \sin A_1 \sin B. \]

Similarly, we may proceed if \( B \) be increased by 90\(^\circ\).

Hence the formulae of Art. 88 are true if either \( A \) or \( B \) be increased by 90\(^\circ\), i.e. they are true if the component angles lie between 0\(^\circ\) and 180\(^\circ\).

Similarly, by putting \( A_2 = 90^\circ + A_1 \), we can prove the truth of the theorems when either or both of the component angles have values between 0\(^\circ\) and 270\(^\circ\).

By proceeding in this way, we see that the theorems are true universally.
90. **Theorem.** To prove that

\[ \sin (A - B) = \sin A \cos B - \cos A \sin B, \]

and \[ \cos (A - B) = \cos A \cos B + \sin A \sin B. \]

Let the revolving line starting from the initial line \( OA \) trace out the angle \( AOB (= A) \), and then, revolving in the opposite direction, trace out the angle \( BOC \), whose magnitude is \( B \). The angle \( AOC \) is therefore \( A - B \).

Take a point \( P \) in the final position of the revolving line, and draw \( PM \) and \( PN \) perpendicular to \( OA \) and \( OB \) respectively; from \( N \) draw \( NQ \) and \( NR \) perpendicular to \( OA \) and \( MP \) respectively.

The angle \( RPN = 90^\circ - \angle PNR = \angle RNB = \angle QON = A \).

Hence

\[ \sin (A - B) = \sin AOC = \frac{MP}{OP} = \frac{MR - PR}{OP} = \frac{QN}{OP} - \frac{PR}{OP} \]

\[ = \frac{QN}{ON} \frac{ON}{OP} - \frac{PR}{PN} \frac{PN}{OP} \]

\[ = \sin A \cos B - \cos RPN \sin B, \]

so that \[ \sin (A - B) = \sin A \cos B - \cos A \sin B. \]

Also

\[ \cos (A - B) = \frac{OM}{OP} = \frac{OQ + QM}{OP} = \frac{OQ}{OP} + \frac{NR}{OP} \]

\[ = \frac{OQ}{ON} \frac{ON}{OP} + \frac{NR}{NP} \frac{NP}{OP} = \cos A \cos B + \sin NPR \sin B, \]

so that \[ \cos (A - B) = \cos A \cos B + \sin A \sin B. \]
91. The proofs of the previous article will be found to apply to angles of any size, provided that due attention be paid to the signs of the quantities involved.

Assuming the truth of the formulae for acute angles, we can shew them to be true universally without drawing any more figures.

For, putting $A_1 = 90^\circ + A$, we have,

\[
\sin (A_1 - B) = \sin [90^\circ + (A - B)] = \cos (A - B) \quad \text{(Art. 70)}
\]

\[
= \cos A \cos B + \sin A \sin B
\]

\[
= \sin A_1 \cos B - \cos A_1 \sin B.
\]

Also \[
\cos (A_1 - B) = \cos [90^\circ + (A - B)] = -\sin (A - B) \quad \text{(Art. 70)}
\]

\[
= -\sin A \cos B + \cos A \sin B
\]

\[
= \cos A_1 \cos B + \sin A_1 \sin B.
\]

Similarly we may proceed if $B$ be increased by $90^\circ$.

Hence the theorem is true for all angles which are not greater than two right angles.

So, by putting $A_2 = 90^\circ + A_1$, we may shew the theorems to be true for all angles less than three right angles, and so on.

Hence, by proceeding in this manner, we may shew that the theorems are true for all angles whatever.

92. The theorems of Arts. 88 and 90, which give respectively the trigonometrical functions of the sum and differences of two angles in terms of the functions of the angles themselves, are often called the Addition and Subtraction Theorems.

93. **Ex. 1.** *Find the values of* $\sin 75^\circ$ *and* $\cos 75^\circ$.

$\sin 75^\circ = \sin (45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

\[
= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}},
\]

and $\cos 75^\circ = \cos (45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

\[
= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.
\]
**Ex. 2.** Prove that \(\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B\),
and
\[
\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B.
\]

By Arts. 88 and 90, we have
\[
\sin (A + B) \sin (A - B) = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)
= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \cdot \sin^2 B
= \sin^2 A - \sin^2 B.
\]

Again, by the same articles, we have
\[
\cos (A + B) \cos (A - B) = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)
= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B
= \cos^2 A - \sin^2 B.
\]

**Ex. 3.** Assuming the formulae for \(\sin (x + y)\) and \(\cos (x + y)\), deduce the formulae for \(\sin (x - y)\) and \(\cos (x - y)\).

We have
\[
\sin x = \sin \{(x - y) + y\} = \sin (x - y) \cos y + \cos (x - y) \sin y \ldots \ldots (1),
\]
and
\[
\cos x = \cos \{(x - y) + y\} = \cos (x - y) \cos y - \sin (x - y) \sin y \ldots \ldots (2).
\]

Multiplying (1) by \(\cos y\) and (2) by \(\sin y\) and subtracting, we have
\[
\sin x \cos y - \cos x \sin y = \sin (x - y) \{\cos^2 y + \sin^2 y\} = \sin (x - y).
\]

Multiplying (1) by \(\sin y\) and (2) by \(\cos y\) and adding, we have
\[
\sin x \sin y + \cos x \cos y = \cos (x - y) \{\cos^2 y + \sin^2 y\} = \cos (x - y).
\]

Hence the two formulae required are proved.

These two formulae are true for all values of the angles, since the formulae from which they are derived are true for all values.

**Examples. XIII.**

1. If \(\sin a = \frac{3}{5}\) and \(\cos \beta = \frac{9}{41}\), find the value of \(\sin (a - \beta)\) and \(\cos (a + \beta)\). Verify by a graph and accurate measurement.

2. If \(\sin a = \frac{45}{53}\) and \(\sin \beta = \frac{33}{65}\), find the values of \(\sin (a - \beta)\) and \(\sin (a + \beta)\).

3. If \(\sin a = \frac{15}{17}\) and \(\cos \beta = \frac{12}{13}\), find the values of \(\sin (a + \beta)\), \(\cos (a - \beta)\), and \(\tan (a + \beta)\). Verify by a graph and accurate measurement.

Prove that
4. \(\cos (45^\circ - A) \cos (45^\circ - B) - \sin (45^\circ - A) \sin (45^\circ - B) = \sin (A + B)\).
5. \[ \sin (45^\circ + A) \cos (45^\circ - B) + \cos (45^\circ + A) \sin (45^\circ - B) = \cos (A - B). \]

6. \[ \frac{\sin (A - B)}{\cos A \cos B} + \frac{\sin (B - C)}{\cos B \cos C} + \frac{\sin (C - A)}{\cos C \cos A} = 0. \]

7. \[ \sin 105^\circ + \cos 105^\circ = \cos 45^\circ. \]

8. \[ \sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ. \]

9. \[ \cos \alpha \cos (\gamma - \alpha) - \sin \alpha \sin (\gamma - \alpha) = \cos \gamma. \]

10. \[ \cos (\alpha + \beta) \cos \gamma - \cos (\beta + \gamma) \cos \alpha = \sin \beta \sin (\gamma - \alpha). \]

11. \[ \sin (n + 1) A \sin (n - 1) A + \cos (n + 1) A \cos (n - 1) A = \cos 2A. \]

12. \[ \sin (n + 1) A \sin (n + 2) A + \cos (n + 1) A \cos (n + 2) A = \cos A. \]

94. From Arts. 88 and 90, we have, for all values of \( A \) and \( B \),

\[ \sin (A + B) = \sin A \cos B + \cos A \sin B, \]

and \[ \sin (A - B) = \sin A \cos B - \cos A \sin B. \]

Hence, by addition and subtraction, we have

\[ \sin (A + B) + \sin (A - B) = 2 \sin A \cos B \ldots \ldots (1), \]

and \[ \sin (A + B) - \sin (A - B) = 2 \cos A \sin B \ldots \ldots (2). \]

From the same articles we have, for all values of \( A \) and \( B \),

\[ \cos (A + B) = \cos A \cos B - \sin A \sin B, \]

and \[ \cos (A - B) = \cos A \cos B + \sin A \sin B. \]

Hence, by addition and subtraction, we have

\[ \cos (A + B) + \cos (A - B) = 2 \cos A \cos B \ldots \ldots (3), \]

and \[ \cos (A - B) - \cos (A + B) = 2 \sin A \sin B \ldots \ldots (4). \]

Put \( A + B = C \), and \( A - B = D \), so that

\[ A = \frac{C + D}{2}, \text{ and } B = \frac{C - D}{2}. \]
On making these substitutions, the relations (1) to (4) become, for all values of $C$ and $D$,

\[
\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \quad \ldots \ldots \text{I},
\]

\[
\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \quad \ldots \ldots \text{II},
\]

\[
\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \quad \ldots \ldots \text{III},
\]

and \quad \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \quad \ldots \ldots \text{IV}.

[The student should carefully notice that the second factor of the right-hand member of IV is $\sin D-C$ and not $\sin \frac{C-D}{2}$.

\]

95. These relations I to IV are extremely important and should be very carefully committed to memory.

On account of their great importance we give a geometrical proof for the case when $C$ and $D$ are acute angles.

Let $AOC$ be the angle $C$ and $AOD$ the angle $D$. Bisect the angle $COD$ by the straight line $OE$. On $OE$ take a point $P$ and draw $QPR$ perpendicular to $OP$ to meet $OC$ and $OD$ in $Q$ and $R$ respectively.

Draw $PL$, $QM$, and $RN$ perpendicular to $OA$, and through $R$ draw $RST$ perpendicular to $PL$ or $QM$ to meet them in $S$ and $T$ respectively.

Since the angle $DOC$ is $C-D$, each of the angles $DOE$ and $EOC$ is $\frac{C-D}{2}$, and also

\[
\angle AOE = \angle AOD + \angle DOE = D + \frac{C-D}{2} = \frac{C+D}{2}.
\]

Since the two triangles $POR$ and $POQ$ are equal in all respects, we have $QQ = OR$, and $PR = PQ$, so that

\[
RQ = 2RP.
\]
Hence $QT = 2PS$, and $RT = 2RS$, i.e. $MN = 2ML$.

Therefore $MQ + NR = TQ + 2LS = 2SP + 2LS = 2LP$.

Also $OM + ON = 2OM + MN = 2OM + 2ML = 2OL$.

Hence $\sin C + \sin D = \frac{MQ}{OQ} + \frac{NR}{OR} = \frac{MQ + NR}{OR}$

$= 2 \frac{LP}{OR} = 2 \frac{LP}{OP} \cdot \frac{OP}{OR} = 2 \sin LOP \cos POR$

$= 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$.

Again, $\sin C - \sin D = \frac{MQ}{OQ} - \frac{NR}{OR} = \frac{MQ - NR}{OR} = \frac{TQ}{OR}$

$= 2 \frac{SP}{OR} = 2 \frac{SP}{RP} \cdot \frac{RP}{OR} = 2 \cos SPR \sin ROP$

$= 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$,

$\left[ \text{for } \angle SPR = 90^\circ - \angle SPO = \angle LOP = \frac{C + D}{2} \right]$.

Also, $\cos C + \cos D = \frac{OM}{OQ} + \frac{ON}{OR} = \frac{OM + ON}{OR}$

$= 2 \frac{LO}{OR} = 2 \frac{LOP}{OP \cdot OR}$

$= 2 \cos LOP \cos POR = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$.
Finally, \( \cos D - \cos C = \frac{ON}{OR} - \frac{OM}{OQ} = \frac{ON - OM}{OR} \)

\( = \frac{MN}{OR} = 2 \frac{SR}{OR} = 2\frac{SR}{PR} \frac{PR}{OR} \)

\( = 2 \sin SPR \cdot \sin POR \)

\( = 2 \sin \frac{C + D}{2} \sin \frac{C - D}{2} . \)

96. The student is strongly urged to make himself perfectly familiar with the formulae of the last article and to carefully practise himself in their application; perfect familiarity with these formulae will considerably facilitate his further progress.

The formulae are very useful, because they change sums and differences of certain quantities into products of certain other quantities, and products of quantities are, as the student probably knows from Algebra, easily dealt with by the help of logarithms.

We subjoin a few examples of their use.

**Ex. 1.** \( \sin 6\theta + \sin 4\theta = 2 \sin \frac{6\theta + 4\theta}{2} \cos \frac{6\theta - 4\theta}{2} = 2 \sin 5\theta \cos \theta . \)

**Ex. 2.** \( \cos 3\theta - \cos 7\theta = 2 \sin \frac{3\theta + 7\theta}{2} \sin \frac{7\theta - 3\theta}{2} = 2 \sin 5\theta \sin 2\theta . \)

**Ex. 3.** \( \frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2 \cos \frac{75^\circ + 15^\circ}{2} \sin \frac{75^\circ - 15^\circ}{2}}{2 \cos \frac{75^\circ + 15^\circ}{2} \cos \frac{75^\circ - 15^\circ}{2}} \)

\( = \frac{2 \cos 45^\circ \sin 30^\circ}{2 \cos 45^\circ \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = .57735...... \)

[This is an example of the simplification given by these formulae; it would be a very long and tiresome process to look out from the tables the values of \( \sin 75^\circ, \sin 15^\circ, \cos 75^\circ, \) and \( \cos 15^\circ, \) and then to perform the division of one long decimal fraction by another.]
EX. 4. Simplify the expression
\[
\frac{(\cos \theta - \cos 3\theta) (\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin \theta) (\cos 4\theta - \cos 6\theta)}.
\]

On applying the formulae of Art. 94, this expression

\[
2 \sin \frac{\theta + 3\theta}{2} \sin \frac{3\theta - \theta}{2} \times 2 \sin \frac{8\theta + 2\theta}{2} \cos \frac{8\theta - 2\theta}{2}
\]

\[
= \frac{4 \cdot \sin 2\theta \sin \theta \cdot \sin 5\theta \cos 3\theta}{4 \cdot \cos 3\theta \sin 2\theta \cdot \sin 5\theta \sin \theta} = 1.
\]

EXAMPLES. XIV.

Prove that

1. \(\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan \theta\).

2. \(\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan \theta\).

3. \(\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A\).

4. \(\frac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \cos 4A \sec 5A\).

5. \(\frac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot (A + B) \cot (A - B)\).

6. \(\frac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \tan (A + B)\).

7. \(\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot \frac{A}{2}\).

8. \(\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A\).

9. \(\frac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \tan (A - B)\).

10. \(\cos (A + B) + \sin (A - B) = 2 \sin (45^\circ + A) \cos (45^\circ + B)\).

11. \(\frac{\cos 3A - \cos A}{\sin 3A - \sin A} + \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{\sin A}{\cos 2A \cos 3A}\).

12. \(\frac{\sin (4A - 2B) + \sin (4B - 2A)}{\cos (4A - 2B) + \cos (4B - 2A)} = \tan (A + B)\).

13. \(\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta \cos 4\theta\).
PRODUCT FORMULAE.

14. \[ \cos 3\theta + 2 \cos 5\theta + \cos 7\theta = \cos 2\theta - \sin 2\theta \tan 3\theta. \]

15. \[ \frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A. \]

16. \[ \frac{\sin (\theta + \phi) - 2 \sin \theta + \sin (\theta - \phi)}{\cos (\theta + \phi) - 2 \cos \theta + \cos (\theta - \phi)} = \tan \theta. \]

17. \[ \frac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}. \]

18. \[ \frac{\sin (A - C) + 2 \sin A + \sin (A + C)}{\sin (B - C) + 2 \sin B + \sin (B + C)} = \frac{\sin A}{\sin B}. \]

19. \[ \frac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A. \]

20. \[ \frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A + B}{2} \cot \frac{A - B}{2}. \]

21. \[ \frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A + B}{2} \cot \frac{A - B}{2}. \]

22. \[ \frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A + B}{2}. \]

23. \[ \frac{\sin A - \sin B}{\cos B - \cos A} = \cot \frac{A + B}{2}. \]

24. \[ \frac{\cos (A + B + C) + \cos (-A + B + C) + \cos (A - B + C) + \cos (A + B - C)}{\sin (A + B + C) + \sin (-A + B + C) - \sin (A - B + C) + \sin (A + B - C)} = \cot B. \]

25. \[ \cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A. \]

26. \[ \cos (-A + B + C) + \cos (A - B + C) + \cos (A + B - C) + \cos (A + B + C) = 4 \cos A \cos B \cos C. \]

27. \[ \sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0. \]

28. \[ \sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ. \]

29. \[ \sin a + \sin 2a + \sin 4a + \sin 5a = 4 \cos \frac{a}{2} \cos \frac{3a}{2} \sin 3a. \]

Simplify

30. \[ \cos \left( \theta + \left( n - \frac{3}{2} \right) \phi \right) - \cos \left( \theta + \left( n + \frac{3}{2} \right) \phi \right). \]

31. \[ \sin \left( \theta + \left( n - \frac{1}{2} \right) \phi \right) + \sin \left( \theta + \left( n + \frac{1}{2} \right) \phi \right). \]

L. T.
Again, by Art. 90,
\[
\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}
\]

\[
= \frac{\sin A \sin B}{\cos A \cos B}, \text{ by dividing as before.}
\]

\[
\therefore \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.
\]

99. The formulae of the preceding article may be obtained geometrically from the figures of Arts. 88 and 90.

(1) Taking the figure of Art. 88, we have
\[
\tan(A + B) = \frac{MP}{OM} = \frac{QN + RP}{OQ - RN}
\]

\[
= \frac{QN + RP}{OQ} + \frac{RP}{OQ} = \frac{\tan A + \frac{RP}{OQ}}{1 - \frac{RN}{OQ}} + \frac{\frac{RP}{OQ}}{\frac{RN}{OQ}}.
\]

But, since the angles \(RPN\) and \(QON\) are equal, the triangles \(RPN\) and \(QON\) are similar, so that
\[
\frac{RP}{PN} = \frac{OQ}{ON},
\]

and therefore
\[
\frac{RP}{OQ} = \frac{PN}{ON} = \tan B.
\]

Hence
\[
\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.
\]

(2) Taking the figure of Art. 90, we have
\[
\tan(A - B) = \frac{MP}{OM} = \frac{QN - PR}{OQ + NR}
\]

\[
= \frac{QN - PR}{OQ} - \frac{PR}{OQ} = \frac{\tan A - \frac{PR}{OQ}}{1 + \frac{NR}{OQ}} - \frac{\frac{PR}{OQ}}{\frac{NR}{OQ}}.
\]
TANGENT OF THE SUM OF TWO ANGLES.

But, since the angles $RPN$ and $NOQ$ are equal, we have $\frac{RP}{PN} = \frac{OQ}{ON}$, and therefore

$$\frac{PR}{OQ} = \frac{PN}{ON} = \tan B.$$

Hence

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan RPN \tan B} = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

100. As particular cases of the preceding formulae, we have, by putting $B$ equal to $45^\circ$,

$$\tan (A + 45^\circ) = \frac{\tan A + 1}{1 - \tan A} = \frac{1 + \tan A}{1 - \tan A},$$

and

$$\tan (A - 45^\circ) = \frac{\tan A - 1}{1 + \tan A}.$$

Similarly, as in Art. 98, we may prove that

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

and

$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

101. Ex. 1. $\tan 75^\circ = \tan (45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$$= 2 + 1.73205... = 3.73205...$$

Ex. 2. $\tan 15^\circ = \tan (45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$$= 2 - 1.73205... = 0.26795...$$
EXEMPLARY XVI.

1. If \( \tan A = \frac{1}{2} \) and \( \tan B = \frac{1}{3} \), find the values of \( \tan (2A + B) \) and \( \tan (2A - B) \). Verify by a graph and accurate measurement.

2. If \( \tan A = \frac{\sqrt{3}}{4 - \sqrt{3}} \) and \( \tan B = \frac{\sqrt{3}}{4 + \sqrt{3}} \), prove that \( \tan (A - B) = \frac{3}{75} \).

3. If \( \tan A = \frac{n}{n + 1} \) and \( \tan B = \frac{1}{2n + 1} \), find \( \tan (A + B) \).

4. If \( \tan A = \frac{5}{6} \) and \( \tan \beta = \frac{1}{11} \), prove that \( A + \beta = \frac{\pi}{4} \). Verify by a graph and accurate measurement.

5. \( \tan \left( \frac{\pi}{4} + \theta \right) \times \tan \left( \frac{3\pi}{4} + \theta \right) = -1 \).

6. \( \cot \left( \frac{\pi}{4} + \theta \right) \cot \left( \frac{\pi}{4} - \theta \right) = 1 \).

7. \( 1 + \tan A \tan \frac{A}{2} = \tan A \cot \frac{A}{2} - 1 = \sec A \).

102. As further examples of the use of the formulae of the present chapter we shall find the general value of the angle which has a given sine, cosine, or tangent. This has been already found in Arts. 82—84.

Find the general value of all angles having a given sine.

Let \( \alpha \) be any angle having the given sine, and \( \theta \) any other angle having the same sine.

We have then to find the most general value of \( \theta \) which satisfies the equation

\[
\sin \theta = \sin \alpha,
\]

i.e. \( \sin \theta - \sin \alpha = 0 \).

This may be written

\[
2 \cos \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2} = 0,
\]
and it is therefore satisfied by
\[ \cos \frac{\theta + \alpha}{2} = 0, \text{ and by } \sin \frac{\theta - \alpha}{2} = 0, \]
i.e. by
\[ \frac{\theta + \alpha}{2} = \text{any odd multiple of } \frac{\pi}{2}, \]
and by
\[ \frac{\theta - \alpha}{2} = \text{any multiple of } \pi, \]
i.e. by
\[ \theta = -\alpha + \text{any odd multiple of } \pi \ldots \ldots (1), \]
and
\[ \theta = \alpha + \text{any even multiple of } \pi \ldots \ldots (2), \]
i.e. \( \theta \) must \( = (-1)^n \alpha + n\pi \), where \( n \) is any positive or negative integer.

For, when \( n \) is odd, this expression agrees with (1), and, when \( n \) is even, it agrees with (2).

103. Find the general value of all angles having the same cosine.

The equation we have now to solve is
\[ \cos \theta = \cos \alpha, \]
i.e.
\[ \cos \alpha - \cos \theta = 0, \]
i.e.
\[ 2 \sin \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2} = 0, \]
and it is therefore satisfied by
\[ \sin \frac{\theta + \alpha}{2} = 0, \text{ and by } \sin \frac{\theta - \alpha}{2} = 0, \]
i.e. by
\[ \frac{\theta + \alpha}{2} = \text{any multiple of } \pi, \]
and by
\[ \frac{\theta - \alpha}{2} = \text{any multiple of } \pi, \]
i.e. by \[ \theta = -\alpha + \text{any multiple of } 2\pi, \]
and by \[ \theta = \alpha + \text{any multiple of } 2\pi. \]

Both these sets of values are included in the solution \( \theta = 2n\pi \pm \alpha \), where \( n \) is any positive or negative integer.

104. **Find the general value of all angles having the same tangent.**

The equation we have now to solve is

\[ \tan \theta - \tan \alpha = 0, \]

*i.e.*

\[ \sin \theta \cos \alpha - \cos \theta \sin \alpha = 0, \]

*i.e.*

\[ \sin (\theta - \alpha) = 0. \]

\[ \therefore \] \( \theta - \alpha = \text{any multiple of } \pi \]

\[ = n\pi, \text{ where } n \text{ is any positive or negative integer,} \]

so that the most general solution is \( \theta = n\pi + \alpha. \)

**EXAMPLES. XVI (a).**

1. Construct the acute angles whose tangents are \( \frac{1}{3} \) and \( \frac{1}{2} \), and verify by measurement that their sum is \( 45^\circ \).

2. The tangents of two acute angles are respectively 3 and 2; show by a graph that the tangent of their difference is \( \frac{1}{7} \).

3. The sine of one acute angle is \( 0.6 \) and the cosine of another is \( 0.5 \). Show graphically, and also by calculation, that the sine of their difference is \( 0.39 \) nearly.

4. Draw the positive angle whose cosine is \( 0.4 \) and show, both by measurement and calculation, that the sine and cosine of an angle which exceeds it by \( 45^\circ \) are \( 0.93 \) and \( -0.365 \) nearly.

5. Draw the acute angle whose tangent is 7 and the acute angle whose sine is \( 0.7 \); and show, both by measurement and calculation, that the sine of their difference is approximately \( 0.61 \).
CHAPTER VIII.

THE TRIGONOMETRICAL RATIOS OF MULTIPLE AND 
SUBMULTIPLE ANGLES.

105. To find the trigonometrical ratios of an angle $2A$
in terms of those of the angle $A$.

If in the formulae of Art. 88 we put $B = A$, we have

$$\sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A,$$

$$\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

$$ = (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A,$$

and also

$$\cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1 ;$$

and

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A} = \frac{2 \tan A}{1 - \tan^2 A} .$$

Now the formulae of Art. 88 are true for all values of
$A$ and $B$; hence any formulae derived from them are true
for all values of the angles.

In particular the above formulae are true for all values
of $A$. 
106. An independent geometrical proof of the formulae of the preceding article may be given for values of $A$ which are less than a right angle.

Let $QCP$ be the angle $2A$.

With centre $C$ and radius $CP$ describe a circle, and let $QC$ meet it again in $O$.

Join $OP$ and $PQ$, and draw $PN$ perpendicular to $OQ$.

By Eucl. III. 20, the angle
\[ QOP = \frac{1}{2} \angle QCP = A, \]
and the angle $\angle NOP = \angle QOP = A$.

Hence
\[ \sin 2A = \frac{NP}{CP} = \frac{2NP}{2CQ} = \frac{2NP}{OQ} = \frac{2NP}{OP} \cdot \frac{OP}{OQ} \]
\[ = 2 \sin NOP \cos POQ, \text{ since } OPQ \text{ is a right angle,} \]
\[ = 2 \sin A \cos A; \]
also
\[ \cos 2A = \frac{CN}{CP} = \frac{2CN}{OQ} = \frac{(OC + CN) - \overrightarrow{OC} - \overrightarrow{CN}}{OQ} \]
\[ = \frac{ON - NQ}{OQ} = \frac{ON}{OP} \cdot \frac{OP}{OQ} - \frac{NQ}{PQ} \cdot \frac{PQ}{OQ} \]
\[ = \cos^2 A - \sin^2 A; \]
and
\[ \tan 2A = \frac{NP}{CN} = \frac{2NP}{ON - NQ} = \frac{2NP}{ON} \cdot \frac{ON}{PN} \cdot \frac{PN}{ON} \]
\[ = \frac{2 \tan A}{1 - \tan^2 A}. \]
Ex. To find the values of $\sin 15^\circ$ and $\cos 15^\circ$.

Let the angle $2A$ be $30^\circ$, so that $A$ is $15^\circ$.
Let the radius $CP$ be $2a$, so that we have

$$CN = 2a \cos 30^\circ = a\sqrt{3},$$
and

$$NP = 2a \sin 30^\circ = a.$$

Hence

$$ON = OC + CN = a (2 + \sqrt{3}),$$
and

$$NQ = CQ - CN = a (2 - \sqrt{3}).$$

$$\therefore OP^2 = ON \cdot OQ = a (2 + \sqrt{3}) \times 4a$$
(Euc. vi. 8),

so that

$$OP = a\sqrt{2 (\sqrt{3} + 1)},$$
and

$$PQ^2 = QN \cdot OQ = a (2 - \sqrt{3}) \times 4a,$$
so that

$$PQ = a\sqrt{2 (\sqrt{3} - 1)}.$$

Hence

$$\sin 15^\circ = \frac{PQ}{OQ} = \frac{\sqrt{2 (\sqrt{3} - 1)}}{4} = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

and

$$\cos 15^\circ = \frac{OP}{OQ} = \frac{\sqrt{2 (\sqrt{3} + 1)}}{4} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

107. To find the trigonometrical functions of $3A$ in terms of those of $A$.

By Art. 88, putting $B$ equal to $2A$, we have

$$\sin 3A = \sin (A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$
$$= \sin A (1 - 2 \sin^2 A) + \cos A \cdot 2 \sin A \cos A,$$

by Art. 105,

$$= \sin A (1 - 2 \sin^2 A) + 2 \sin A (1 - \sin^2 A).$$

Hence

$$\sin 3A = 3 \sin A - 4 \sin^3 A \ldots \ldots (1).$$

So

$$\cos 3A = \cos (A + 2A) = \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A (2 \cos^2 A - 1) - \sin A \cdot 2 \sin A \cos A$$
$$= \cos A (2 \cos^2 A - 1) - 2 \cos A (1 - \cos^2 A).$$

Hence

$$\cos 3A = 4 \cos^3 A - 3 \cos A \ldots \ldots (2).$$
Also   \[\tan 3A = \tan (A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}\]

\[= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \frac{2 \tan A}{1 - \tan^2 A}} = \frac{\tan A (1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A}.
\]

Hence   \[\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.\]

[The student may find it difficult to remember, and distinguish between, the formulae (1) and (2), which bear a general resemblance to one another, but have their signs in a different order. If in doubt, he may always verify his formula by testing it for a particular case, e.g. by putting \(A = 30^\circ\) for formula (1), and by putting \(A = 0^\circ\) for formula (2).]

108. By a process similar to that of the last article, the trigonometrical ratios of any higher multiples of \(\theta\) may be expressed in terms of those of \(\theta\). The method is however long and tedious. In a later chapter better methods will be pointed out.

As an example, let us express \(\cos 5\theta\) in terms of \(\cos \theta\). We have

\[\cos 5\theta = \cos (3\theta + 2\theta)\]

\[= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta\]

\[= (4 \cos^3 \theta - 3 \cos \theta) (2 \cos^2 \theta - 1)\]

\[= (8 \cos^5 \theta - 10 \cos^3 \theta + 3 \cos \theta) - 2 \cos \theta \cdot \sin^2 \theta (3 - 4 \sin^2 \theta)\]

\[= (8 \cos^5 \theta - 10 \cos^3 \theta + 3 \cos \theta) - 2 \cos \theta (1 - \cos^2 \theta) (4 \cos^2 \theta - 1)\]

\[= (8 \cos^5 \theta - 10 \cos^3 \theta + 3 \cos \theta) - 2 \cos \theta (5 \cos^2 \theta - 4 \cos^4 \theta - 1)\]

\[= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.\]
EXAMPLES. XVII.

1. Find the value of \( \sin 2\alpha \) when
   
   (1) \( \cos \alpha = \frac{3}{5} \), (2) \( \sin \alpha = \frac{12}{13} \), and (3) \( \tan \alpha = \frac{16}{63} \).

2. Find the value of \( \cos 2\alpha \) when
   
   (1) \( \cos \alpha = \frac{15}{17} \), (2) \( \sin \alpha = \frac{4}{5} \), and (3) \( \tan \alpha = \frac{5}{12} \).

   Verify by a graph and accurate measurement.

3. If \( \tan \theta = \frac{b}{a} \), find the value of \( a \cos 2\theta + b \sin 2\theta \).

   Prove that

4. \( \frac{\sin 2A}{1 + \cos 2A} = \tan A \).

5. \( \frac{\sin 2A}{1 - \cos 2A} = \cot A \).

6. \( \frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A \).

7. \( \tan A + \cot A = 2 \csc 2A \).

8. \( \tan A - \cot A = -2 \cot 2A \).

9. \( \csc 2A + \cot 2A = \cot A \).

10. \( \frac{1 - \cos A + \cos B - \cos (A + B)}{1 + \cos A - \cos B - \cos (A + B)} = \tan \frac{A}{2} \cot \frac{B}{2} \).

11. \( \frac{\cos A}{1 + \sin A} = \tan \left( \frac{45^\circ + A}{2} \right) \).

12. \( \frac{\sec 8A - 1}{\sec 4A - 1} = \tan \frac{8A}{2} \).

13. \( \frac{1 + \tan^2 (45^\circ - A)}{1 - \tan^2 (45^\circ - A)} = \csc 2A \).

14. \( \frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}} \).

15. \( \frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A + B) \).

16. \( \tan \left( \frac{\pi}{4} + \theta \right) - \tan \left( \frac{\pi}{4} - \theta \right) = 2 \tan 2\theta \).

17. \( \frac{\cos A + \sin A}{\cos A - \sin A} = \frac{\cos A - \sin A}{\cos A + \sin A} = 2 \tan 2A \).

18. \( \cot (A + 15^\circ) - \tan (A - 15^\circ) = \frac{4 \cos 2A}{1 + 2 \sin 2A} \).
19. \[ \frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta. \]

20. \[ \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \frac{\theta}{2}. \]

21. \[ \frac{\sin (n+1)A - \sin (n-1)A}{\cos (n+1)A + 2 \cos nA + \cos (n-1)A} = \tan \frac{A}{2}. \]

22. \[ \frac{\sin (n+1)A + 2 \sin nA + \sin (n-1)A}{\cos (n-1)A - \cos (n+1)A} = \cot \frac{A}{2}. \]

23. \[ \sin (2n+1)A \sin A = \sin^2 (n+1)A - \sin^2 nA. \]

24. \[ \frac{\sin (A + 3B) + \sin (3A + B)}{\sin 2A + \sin 2B} = 2 \cos (A + B). \]

25. \[ \sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}. \]

26. \[ \tan 2A = (\sec 2A + 1) \sqrt{\sec^2 A - 1}. \]

27. \[ \cos^3 \theta + 3 \cos \theta = 4 (\cos^4 \theta - \sin^4 \theta). \]

28. \[ 1 + \cos^2 \theta = 2 (\cos^4 \theta + \sin^4 \theta). \]

29. \[ \sec^2 A (1 + \sec 2A) = 2 \sec 2A. \]

30. \[ \csc A - 2 \cot 2A \cos A = 2 \sin A. \]

31. \[ \cot A = \frac{1}{2} \left( \cot \frac{A}{2} - \tan \frac{A}{2} \right). \]

32. \[ \sin a \sin (60^\circ - a) \sin (60^\circ + a) = \frac{1}{4} \sin 3a. \]

33. \[ \cos a \cos (60^\circ - a) \cos (60^\circ + a) = \frac{1}{4} \cos 3a. \]

34. \[ \cot a + \cot (60^\circ + a) - \cot (60^\circ - a) = 3 \cot 3a. \]

35. \[ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}. \]

36. \[ \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}. \]

37. \[ \cos 4a = 1 - 8 \cos^2 a + 8 \cos^4 a. \]

38. \[ \sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A. \]

39. \[ \cos 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1. \]

40. \[ \tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A. \]

41. \[ \frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1) (2 \cos 2\theta - 1) (2 \cos 2^2 \theta - 1) \]

...... (2 \cos 2^{n-1} \theta - 1).
Submultiple angles.

109. Since the relations of Art. 105 are true for all values of the angle \( A \), they will be true if instead of \( A \) we substitute \( \frac{A}{2} \), and therefore if instead of \( 2A \) we put \( 2 \cdot \frac{A}{2} \), i.e. \( A \).

Hence we have the relations

\[
\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \quad \text{.................. (1)},
\]

\[
\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \quad \text{.......................... (2)},
\]

\[
\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \quad \text{.................. (3)}.
\]

From (1), we also have

\[
\sin A = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}} \quad \text{.......................... (4)},
\]

\[
= \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}, \quad \text{by dividing numerator and denominator by } \cos^2 \frac{A}{2}.
\]
TRIGONOMETRY.

So
\[
\cos A = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}} = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}.
\]

110. To express the trigonometrical ratios of the angle \(\frac{A}{2}\) in terms of \(\cos A\).

From equation (2) of the last article, we have
\[
\cos A = 1 - 2 \sin^2 \frac{A}{2},
\]
so that
\[
2 \sin^2 \frac{A}{2} = 1 - \cos A,
\]
and therefore
\[
\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1).
\]

Again,
\[
\cos A = 2 \cos^2 \frac{A}{2} - 1,
\]
so that
\[
2 \cos^2 \frac{A}{2} = 1 + \cos A,
\]
and therefore
\[
\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2).
\]

Hence,
\[
\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3).
\]
RATIOS OF $\frac{A}{2}$ IN TERMS OF $\cos A$. 113

111. In each of the preceding formulae it will be noted that there is an ambiguous sign. In any particular case the proper sign can be determined as the following examples will shew.

Ex. 1. Given $\cos 45^\circ = \frac{1}{\sqrt{2}}$, find the values of $\sin 22\frac{1}{2}^\circ$ and $\cos 22\frac{1}{2}^\circ$.

The equation (1) of the last article gives, by putting $A$ equal to $45^\circ$,

$$\sin 22\frac{1}{2}^\circ = \pm \sqrt{\frac{1 - \cos 45^\circ}{2}} = \pm \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} = \pm \sqrt{\frac{2 - \sqrt{2}}{4}}$$

$$= \pm \frac{1}{2} \sqrt{2 - \sqrt{2}}.$$

Now $\sin 22\frac{1}{2}^\circ$ is necessarily positive, so that the upper sign must be taken.

Hence

$$\sin 22\frac{1}{2}^\circ = \frac{1}{2} \sqrt{2 - \sqrt{2}}.$$

So

$$\cos 22\frac{1}{2}^\circ = \pm \sqrt{\frac{1 + \cos 15^\circ}{2}} = \pm \sqrt{\frac{2 + \sqrt{2}}{4}} = \pm \frac{1}{2} \sqrt{2 + \sqrt{2}};$$

also $\cos 22\frac{1}{2}^\circ$ is positive;

$$\therefore \cos 22\frac{1}{2}^\circ = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

Ex. 2. Given $\cos 330^\circ = \frac{\sqrt{3}}{2}$, find the values of $\sin 165^\circ$ and $\cos 165^\circ$.

The equation (1) gives

$$\sin 165^\circ = \pm \sqrt{\frac{1 - \cos 330^\circ}{2}} = \pm \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \pm \sqrt{\frac{4 - 2\sqrt{3}}{8}}$$

$$= \pm \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

Also

$$\cos 165^\circ = \pm \sqrt{\frac{1 + \cos 330^\circ}{2}} = \pm \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \pm \sqrt{\frac{4 + 2\sqrt{3}}{8}}$$

$$= \pm \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

L. T. 8
Now 165° lies between 90° and 180°, so that, by Art. 52, its sine is positive and its cosine is negative.

Hence \[ \sin 165^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}, \]
and \[ \cos 165^\circ = -\frac{\sqrt{3} + 1}{2\sqrt{2}}. \]

From the above examples it will be seen that, when the angle \( A \) and its cosine are given, the ratios for the angle \( \frac{A}{2} \) may be determined without any ambiguity of sign.

When, however, only \( \cos A \) is given, there is an ambiguity in finding \( \sin \frac{A}{2} \) and \( \cos \frac{A}{2} \). The explanation of this ambiguity is given in the next article.

**112. To explain why there is ambiguity when \( \cos \frac{A}{2} \) and \( \sin \frac{A}{2} \) are found from the value of \( \cos A \).**

We know that, if \( n \) be any integer,

\[ \cos A = \cos (2n\pi \pm A) = k \text{ (say)}. \]

Hence any formula which gives us \( \cos \frac{A}{2} \) in terms of \( k \), should give us also the cosine of \( \frac{2n\pi \pm A}{2} \).

Now \[ \cos \frac{2n\pi \pm A}{2} = \cos \left( n\pi \pm \frac{A}{2} \right) \]

\[ = \cos n\pi \cos \frac{A}{2} \mp \sin n\pi \sin \frac{A}{2} = \cos n\pi \cos \frac{A}{2} \]

\[ = \pm \cos \frac{A}{2}, \]

according as \( n \) is even or odd.
Similarly, any formula, giving us \( \sin \frac{A}{2} \) in terms of \( k \), should give us also the sine of \( \frac{2n\pi \pm A}{2} \).

Also,
\[
\sin \frac{2n\pi \pm A}{2} = \sin \left( n\pi \pm \frac{A}{2} \right)
= \sin n\pi \cos \frac{A}{2} \pm \cos n\pi \sin \frac{A}{2} = \pm \cos n\pi \sin \frac{A}{2}
= \pm \sin \frac{A}{2}.
\]

Hence, in each case, we should expect to obtain two values for \( \cos \frac{A}{2} \) and \( \sin \frac{A}{2} \), and this is the number which the formulae of Art. 110 give.

[The student may illustrate this article geometrically by drawing the angles \( \frac{2n\pi \pm A}{2} \), i.e. \( n\pi \pm \frac{A}{2} \). The bounding line for these angles will have four positions, two inclined to the positive direction of the initial line at angles \( \frac{A}{2} \) and \( -\frac{A}{2} \), and two inclined at \( \frac{A}{2} \) and \( -\frac{A}{2} \) to the negative direction of the initial line. It will be clear from the figure that there are two values for \( \cos \frac{A}{2} \) and two for \( \sin \frac{A}{2} \).]

113. To express the trigonometrical ratios of the angle \( \frac{A}{2} \) in terms of \( \sin A \).

From equation (1) of Art. 109, we have
\[
2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A \quad \ldots \ldots (1).
\]

Also,
\[
\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1, \text{ always} \quad \ldots \ldots (2).
\]
First adding these equations, and then subtracting (1) from (2), we have
\[ \sin^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 + \sin A, \]
and
\[ \sin^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 - \sin A; \]
i.e.
\[ \left( \sin \frac{A}{2} + \cos \frac{A}{2} \right)^2 = 1 + \sin A, \]
and
\[ \left( \sin \frac{A}{2} - \cos \frac{A}{2} \right)^2 = 1 - \sin A; \]
so that
\[ \sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \ldots \ldots \ldots \ldots (3), \]
and
\[ \sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \ldots \ldots \ldots \ldots (4). \]

By adding, and then subtracting, we have
\[ 2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A} \ldots \ldots (5), \]
and
\[ 2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A} \ldots \ldots (6). \]

The other ratios of \( \frac{A}{2} \) are then easily obtained.

114. In each of the formulae (5) and (6) there are two ambiguous signs. In the following examples it is shewn how to determine the ambiguity in any particular case.

\textbf{Ex. 1.} Given that \( \sin 30^\circ \) is \( \frac{1}{2} \), find the values of \( \sin 15^\circ \) and \( \cos 15^\circ \).

Putting \( A = 30^\circ \), we have from relations (3) and (4),
RATIOS OF $\frac{A}{2}$ IN TERMS OF SIN $A$.  

\[
\sin 15^\circ + \cos 15^\circ = \pm \frac{\sqrt{1 + \sin 30^\circ}}{\sqrt{2}} = \pm \frac{\sqrt{3}}{\sqrt{2}},
\]

\[
\sin 15^\circ - \cos 15^\circ = \pm \frac{\sqrt{1 - \sin 30^\circ}}{\sqrt{2}} = \pm \frac{1}{\sqrt{2}}.
\]

Now $\sin 15^\circ$ and $\cos 15^\circ$ are both positive, and $\cos 15^\circ$ is greater than $\sin 15^\circ$. Hence the expressions $\sin 15^\circ + \cos 15^\circ$ and $\sin 15^\circ - \cos 15^\circ$ are respectively positive and negative.

Hence the above two relations should be

\[
\sin 15^\circ + \cos 15^\circ = \frac{\sqrt{3}}{\sqrt{2}},
\]

and

\[
\sin 15^\circ - \cos 15^\circ = -\frac{1}{\sqrt{2}}.
\]

Hence $\sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$, and $\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$.

**Ex. 2.** Given that $\sin 570^\circ$ is equal to $-\frac{1}{2}$, find the values of $\sin 235^\circ$ and $\cos 285^\circ$.

Putting $A$ equal to $570^\circ$, we have

\[
\sin 235^\circ + \cos 285^\circ = \pm \frac{\sqrt{1 + \sin 570^\circ}}{\sqrt{2}} = \pm \frac{1}{\sqrt{3}},
\]

and

\[
\sin 285^\circ - \cos 285^\circ = \pm \frac{\sqrt{1 - \sin 570^\circ}}{\sqrt{2}} = \pm \frac{\sqrt{3}}{2}.
\]

Now $\sin 285^\circ$ is negative, $\cos 285^\circ$ is positive, and the former is numerically greater than the latter, as may be seen by a figure.

Hence $\sin 235^\circ + \cos 285^\circ$ is negative, and $\sin 285^\circ - \cos 285^\circ$ is also negative.

\[
\therefore \sin 235^\circ + \cos 285^\circ = -\frac{1}{\sqrt{3}},
\]

and

\[
\sin 235^\circ - \cos 285^\circ = -\frac{\sqrt{3}}{\sqrt{2}}.
\]

Hence $\sin 235^\circ = -\frac{\sqrt{3} + 1}{2\sqrt{2}}$, and

\[
\cos 285^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}.
\]
**115. To explain why there is ambiguity when \( \sin \frac{A}{2} \) and \( \cos \frac{A}{2} \) are found from the value of \( \sin A \).

We know that, if \( n \) be any integer,

\[
\sin \left\{ n\pi + (-1)^n A \right\} = \sin A = k \text{ (say).} \quad \text{(Art. 82.)}
\]

Hence any formula which gives us \( \sin \frac{A}{2} \) in terms of \( k \),

should give us also the sine of \( \frac{n\pi + (-1)^n A}{2} \).

First, let \( n \) be even and equal to \( 2m \). Then

\[
\sin \frac{n\pi + (-1)^n A}{2} = \sin \left( m\pi + \frac{A}{2} \right)
\]

\[
= \sin m\pi \cos \frac{A}{2} + \cos m\pi \sin \frac{A}{2} = \cos m\pi \sin \frac{A}{2}
\]

\[
= \pm \sin \frac{A}{2},
\]

according as \( m \) is even or odd.

Secondly, let \( n \) be odd and equal to \( 2p + 1 \).

Then

\[
\sin \frac{n\pi + (-1)^n A}{2} = \sin \frac{2p\pi + \pi - A}{2} = \sin \left[ p\pi + \frac{\pi - A}{2} \right]
\]

\[
= \sin p\pi \cos \frac{\pi - A}{2} + \cos p\pi \sin \frac{\pi - A}{2} = \cos p\pi \cos \frac{A}{2}
\]

\[
= \pm \cos \frac{A}{2},
\]

according as \( p \) is even or odd.

Hence any formula which gives us \( \sin \frac{A}{2} \) in terms of \( \sin A \) should be expected to give us, in addition, the values of

\[-\sin \frac{A}{2}, \cos \frac{A}{2} \text{ and } -\cos \frac{A}{2},\]
i.e. 4 values in all. This is the number of values which we get from the formulae of Art. 113, by giving all possible values to the ambiguities.

In a similar manner it may be shewn that when \( \cos \frac{A}{2} \) is found from \( \sin A \), we should expect 4 values.

[If the angles \( \frac{n\pi}{2} + (-1)^n \frac{A}{2} \), i.e. \( n\frac{\pi}{2} + (-1)^n \frac{A}{2} \), be drawn geometrically for the case when \( \frac{A}{2} \) is an acute angle, it will be found that there are four positions of the bounding line, two in the first quadrant inclined at angles \( \frac{A}{2} \) and \( \frac{\pi}{2} - \frac{A}{2} \) to the initial line, and two in the third quadrant inclined at \( \frac{A}{2} \) and \( \frac{\pi}{2} - \frac{A}{2} \) to the negative direction of the initial line. It will be clear from the figure that we should then expect four values for \( \sin \frac{A}{2} \) and four for \( \cos \frac{A}{2} \). Similarly for any other value of \( \frac{A}{2} \).]

116. In any general case we can shew how the ambiguities in relations (3) and (4) of Art. 113 may be found.

We have

\[
\sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin \frac{A}{2} + \frac{1}{\sqrt{2}} \cos \frac{A}{2} \right)
\]

\[
= \sqrt{2} \left[ \sin \frac{A}{2} \cos \frac{\pi}{4} + \cos \frac{A}{2} \sin \frac{\pi}{4} \right] = \sqrt{2} \sin \left( \frac{\pi}{4} + \frac{A}{2} \right)
\]

The right-hand member of this equation is positive if

\( \frac{\pi}{4} + \frac{A}{2} \) lie between \( 2n\pi \) and \( 2n\pi + \pi \),

i.e. if \( \frac{A}{2} \) lie between \( 2n\pi - \frac{\pi}{4} \) and \( 2n\pi + \frac{3\pi}{4} \).
Hence \( \sin \frac{A}{2} + \cos \frac{A}{2} \) is positive if \( \frac{A}{2} \) lie between 
\[ 2n\pi - \frac{\pi}{4} \] and \( 2n\pi + \frac{3\pi}{4} \; ; \]
it is negative otherwise.

Similarly we can prove that
\[ \sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{2} \sin \left( \frac{A}{2} - \frac{\pi}{4} \right). \]

Therefore \( \sin \frac{A}{2} - \cos \frac{A}{2} \) is positive if
\( \left( \frac{A}{2} - \frac{\pi}{4} \right) \) lie between \( 2n\pi \) and \( 2n\pi + \pi \),
i.e. if \( \frac{A}{2} \) lie between \( 2n\pi + \frac{\pi}{4} \) and \( 2n\pi + \frac{5\pi}{4} \).

It is negative otherwise.

The results of this article are shewn graphically in the following figure.

\[ \text{OA is the initial line, and OP, OQ, OR and OS bisect} \]
the angles in the first, second, third and fourth quadrants respectively.

**Numerical Example.** Within what limits must \( \frac{A}{2} \) lie if

\[
2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}
\]

In this case the formulae of Art. 113 must clearly be

\[
\sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A} \quad \ldots \ldots \quad (1),
\]

and

\[
\sin \frac{A}{2} - \cos \frac{A}{2} = -\sqrt{1 - \sin A} \quad \ldots \ldots \quad (2).
\]

For the addition of these two formulae gives the given formula.

From (1) it follows that the revolving line which bounds the angle \( \frac{A}{2} \) must be between \( OQ \) and \( OR \) or else between \( OR \) and \( OS \).

From (2), it follows that the revolving line must lie between \( OR \) and \( OS \) or else between \( OS \) and \( OP \).

Both these conditions are satisfied only when the revolving line lies between \( OR \) and \( OS \), and therefore the angle \( \frac{A}{2} \) lies between

\[
2n\pi - \frac{3\pi}{4} \quad \text{and} \quad 2n\pi - \frac{\pi}{4}.
\]

**117.** To express the trigonometrical ratios of \( \frac{A}{2} \) in terms of \( \tan A \).

From equation (3) of Art. 109, we have

\[
\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}.
\]

\[
\therefore \quad 1 - \tan^2 \frac{A}{2} = \frac{2}{\tan A} \tan \frac{A}{2}.
\]

Hence

\[
\tan^2 \frac{A}{2} + \frac{2}{\tan A} \tan \frac{A}{2} + \frac{1}{\tan^2 A} = 1 + \frac{1}{\tan^2 A} = \frac{1 + \tan^2 A}{\tan^2 A}.
\]
\[ \therefore \tan \frac{A}{2} + \frac{1}{\tan A} = \pm \frac{\sqrt{1 + \tan^2 A}}{\tan A}. \]

\[ \therefore \tan \frac{A}{2} = \pm \frac{\sqrt{1 + \tan^2 A} - 1}{\tan A} \] ...........(1).

118. The ambiguous sign in equation (1) can only be determined when we know something of the magnitude of \( A \).

**Ex.** Given \( \tan 15^\circ = 2 - \sqrt{3} \), find \( \tan 7\frac{1}{2}^\circ \).

Putting \( A = 15^\circ \) we have, from equation (1) of the last article,

\[ \tan 7\frac{1}{2}^\circ = \pm \frac{\sqrt{1 + (2 - \sqrt{3})^2} - 1}{2 - \sqrt{3}} = \pm \frac{\sqrt{8 - 4\sqrt{3} - 1}}{2 - \sqrt{3}} \] ...........(1).

Now \( \tan 7\frac{1}{2}^\circ \) is positive, so that we must take the upper sign.

Hence

\[ \tan 7\frac{1}{2}^\circ = \frac{\sqrt{6} - \sqrt{2} - 1}{2 - \sqrt{3}} \]

\[ = (\sqrt{6} - \sqrt{2} - 1)(2 + \sqrt{3}) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2 = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1). \]

Since \( \tan 15^\circ = \tan 195^\circ \), the equation which gives us \( \tan \frac{15^\circ}{2} \) in terms of \( \tan 15^\circ \) may be expected to give us \( \tan \frac{195^\circ}{2} \) in terms of \( \tan 195^\circ \). In fact the value obtained from (1) by taking the negative sign before the radical is \( \tan \frac{195^\circ}{2} \).

Hence

\[ \tan \frac{195^\circ}{2} = -\frac{\sqrt{8 - 4\sqrt{3} - 1}}{2 - \sqrt{3}} = \frac{\sqrt{6} - \sqrt{2} - 1}{2 - \sqrt{3}} \]

\[ = (-\sqrt{6} + \sqrt{2} - 1)(2 + \sqrt{3}) = -(\sqrt{3} + \sqrt{2})(\sqrt{2} + 1), \]

so that

\[ -\cot 7\frac{1}{2}^\circ = \tan 97\frac{1}{2}^\circ = -(\sqrt{3} + \sqrt{2})(\sqrt{2} + 1). \]

**119.** To explain why there is ambiguity when \( \tan \frac{A}{2} \) is found from the value of \( \tan A \).

We know, by Art. 84, that, if \( n \) be any integer,

\[ \tan (n\pi + A) = \tan A = k \] (say).
Hence any equation which gives us $\tan \frac{A}{2}$ in terms of $k$ may be expected to give us $\tan \frac{n\pi + A}{2}$ also.

First, let $n$ be even and equal to $2m$.

Then

$$\tan \frac{n\pi + A}{2} = \tan \frac{2m\pi + A}{2} = \tan \left( m\pi + \frac{A}{2} \right)$$

$$= \tan \frac{A}{2}, \text{ as in Art. 84.}$$

Secondly, let $n$ be odd and equal to $2p + 1$.

Then

$$\tan \frac{n\pi + A}{2} = \tan \frac{(2p + 1)\pi + A}{2}$$

$$= \tan \left( p\pi + \frac{\pi + A}{2} \right) = \tan \frac{\pi + A}{2} \quad \text{(Art. 84)}$$

$$= -\cot \frac{A}{2}. \quad \text{(Art. 70.)}$$

Hence the formula which gives us the value of $\tan \frac{A}{2}$ should be expected to give us also the value of $-\cot \frac{A}{2}$.

An illustration of this is seen in the example of the last article.

**EXAMPLES. XVIII.**

1. If $\sin \theta = \frac{1}{2}$ and $\sin \phi = \frac{1}{3}$, find the values of $\sin (\theta + \phi)$ and $\sin (2\theta + 2\phi)$.

2. The tangent of an angle is 2·4. Find its cosecant, the cosecant of half the angle, and the cosecant of the supplement of double the angle.
3. If \( \cos \alpha = \frac{11}{61} \) and \( \sin \beta = \frac{4}{5} \), find the values of \( \sin^2 \frac{\alpha - \beta}{2} \) and \( \cos^2 \frac{\alpha + \beta}{2} \), the angles \( \alpha \) and \( \beta \) being positive acute angles.

4. If \( \cos \alpha = \frac{3}{5} \) and \( \cos \beta = \frac{4}{5} \), find the value of \( \cos \frac{\alpha + \beta}{2} \), the angles \( \alpha \) and \( \beta \) being positive acute angles.

5. Given \( \sec \theta = 1\frac{1}{2} \), find \( \tan \frac{\theta}{2} \) and \( \tan \theta \). Verify by a graph.

6. If \( \cos A = -0.28 \), find the value of \( \tan \frac{A}{2} \), and explain the resulting ambiguity.

7. Find the values of (1) \( \sin 7\frac{1}{2}^\circ \), (2) \( \cos 7\frac{1}{2}^\circ \), (3) \( \tan 22\frac{1}{2}^\circ \), and (4) \( \tan 11\frac{1}{4}^\circ \).

8. If \( \sin \theta + \sin \phi = a \) and \( \cos \theta + \cos \phi = b \), find the value of \( \tan \frac{\theta - \phi}{2} \). Prove that

9. \( (\cos \alpha + \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \cos^2 \frac{\alpha + \beta}{2} \).

10. \( (\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \frac{\alpha - \beta}{2} \).

11. \( (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \sin^2 \frac{\alpha - \beta}{2} \).

12. \( \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \).

13. \( \cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \).

14. \( \sec \left( \frac{\pi}{4} + \theta \right) \sec \left( \frac{\pi}{4} - \theta \right) = 2 \sec 2\theta \).

15. \( \tan \left( 45^\circ + \frac{A}{2} \right) = \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A \).

16. \( \sin^2 \left( \frac{\pi}{8} + \frac{A}{2} \right) - \sin^2 \left( \frac{\pi}{8} - \frac{A}{2} \right) = \frac{1}{\sqrt{2}} \sin A \).

17. \( \cos^2 \alpha + \cos^2 (\alpha + 120^\circ) + \cos^2 (\alpha - 120^\circ) = \frac{3}{2} \).

18. \( \cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{8}{2} \).
19. \( \sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} = \frac{3}{2} \).

20. \( \cos 2\theta \cos 2\phi + \sin^2 (\theta - \phi) - \sin^2 (\theta + \phi) = \cos (2\theta + 2\phi) \).

21. \((\tan 4A + \tan 2A)(1 - \tan^2 3A \tan^2 A) = 2 \tan 3A \sec^2 A \).

22. \( \left(1 + \tan \frac{a}{2} - \sec \frac{a}{2}\right) \left(1 + \tan \frac{a}{2} + \sec \frac{a}{2}\right) = \sin a \sec^2 \frac{a}{2} \).

Find the proper signs to be applied to the radicals in the three following formulae.

23. \(2 \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}, \) when \( \frac{A}{2} = 278^\circ \).

24. \(2 \sin \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}, \) when \( \frac{A}{2} = \frac{19\pi}{11} \).

25. \(2 \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}, \) when \( \frac{A}{2} = -140^\circ \).

26. If \( A = 340^\circ \), prove that

\[
2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A},
\]

and

\[
2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}.
\]

27. If \( A = 460^\circ \), prove that

\[
2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}.
\]

28. If \( A = 580^\circ \), prove that

\[
2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}.
\]

29. Within what respective limits must \( \frac{A}{2} \) lie when

\[
(1) \quad 2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A},
\]

\[
(2) \quad 2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A},
\]

\[
(3) \quad 2 \sin \frac{A}{2} = + \sqrt{1 + \sin A} - \sqrt{1 - \sin A},
\]

and

\[
(4) \quad 2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.
\]
30. In the formula
\[ 2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}, \]
find within what limits \( \frac{A}{2} \) must lie when

(1) the two positive signs are taken,
(2) the two negative " " " "
and (3) the first sign is negative and the second positive.

31. Prove that the sine is algebraically less than the cosine for any angle between \( 2n\pi - \frac{3\pi}{4} \) and \( 2n\pi + \frac{\pi}{4} \) where \( n \) is any integer.

32. If \( \sin \frac{A}{3} \) be determined from the equation
\[ \sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}, \]
prove that we should expect to obtain also the values of
\[ \sin \frac{\pi - A}{3} \text{ and } - \sin \frac{\pi + A}{3}. \]
Give also a geometrical illustration.

33. If \( \cos \frac{A}{3} \) be found from the equation
\[ \cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}, \]
prove that we should expect to obtain also the values of
\[ \cos \frac{2\pi - A}{3} \text{ and } \cos \frac{2\pi + A}{3}. \]
Give also a geometrical illustration.

120. By the use of the formulae of the present chapter we can now find the trigonometrical ratios of some important angles.

To find the trigonometrical functions of an angle of 18°.
Let \( \theta \) stand for 18°, so that 2\( \theta \) is 36° and 3\( \theta \) is 54°.
Hence \( 2\theta = 90° - 3\theta \),
and therefore
\[ \sin 2\theta = \sin (90° - 3\theta) = \cos 3\theta. \]
\[ \therefore 2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta \ (\text{Arts. 105 and 107}). \]
Hence, either \( \cos \theta = 0 \), which gives \( \theta = 90^\circ \), or
\[
2 \sin \theta = 4 \cos^2 \theta - 3 = 1 - 4 \sin^2 \theta.
\]
\[\therefore 4 \sin^2 \theta + 2 \sin \theta = 1.\]
By solving this quadratic equation, we have
\[
\sin \theta = \frac{\pm \sqrt{5} - 1}{4}.
\]
In our case \( \sin \theta \) is necessarily a positive quantity. Hence we take the upper sign, and have
\[
\sin 18^\circ = \frac{\sqrt{5} - 1}{4}.
\]
Hence
\[
\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ} = \sqrt{1 - \frac{6 - 2\sqrt{5}}{16}} = \sqrt{\frac{10 + 2\sqrt{5}}{16}} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.
\]
The remaining trigonometrical ratios of \( 18^\circ \) may be now found.
Since \( 72^\circ \) is the complement of \( 18^\circ \), the values of the ratios for \( 72^\circ \) may be obtained by the use of Art. 69.

121. To find the trigonometrical functions of an angle of \( 36^\circ \).
Since \( \cos 2\theta = 1 - 2 \sin^2 \theta \), (Art. 105),
\[\therefore \cos 36^\circ = 1 - 2 \sin^2 18^\circ = 1 - 2 \left( \frac{6 - 2\sqrt{5}}{16} \right) = 1 - \frac{3 - \sqrt{5}}{4},\]
so that \( \cos 36^\circ = \frac{\sqrt{5} + 1}{4} \).
Hence
\[
\sin 36^\circ = \sqrt{1 - \cos^2 36^\circ} = \sqrt{1 - \frac{6 + 2\sqrt{5}}{16}} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.
\]
The remaining trigonometrical functions of 36° may now be found.

Also, since 54° is the complement of 36°, the values of the functions for 54° may be found by the help of Art. 69.

122. The value of \( \sin 18° \) and \( \cos 36° \) may also be found geometrically as follows.

Let \( ABC \) be a triangle constructed, as in Euc. iv. 10, so that each of the angles \( B \) and \( C \) is double of the angle \( A \). Then

\[
180° = A + B + C = A + 2A + 2A,
\]
so that \( A = 36°. \)

Hence, if \( AD \) be drawn perpendicular to \( BC \), we have

\[
\angle BAD = 18°.
\]

By Euclid's construction we know that \( BC \) is equal to \( AX \) where \( X \) is a point on \( AB \), such that

\[
AB \cdot BX = AX^2.
\]

Let \( AB = a \), and \( AX = x. \)

This relation then gives

\[
\frac{a}{2} (a - x) = x^2,
\]

\[
\text{i.e.} \quad x^2 + ax = a^2,
\]

\[
\text{i.e.} \quad x = a \frac{\sqrt{5} - 1}{2}.
\]

Hence \( \sin 18° = \sin BAD = \frac{BD}{BA} = \frac{1}{2} \frac{BC}{BA} \)

\[
= \frac{1}{2} \frac{x}{a} = \frac{\sqrt{5} - 1}{4}.
\]
Again, (by Euc. iv. 10), we know that $AX$ and $XC$ are equal; hence, if $XL$ be perpendicular to $AC$, then $L$ bisects $AC$.

Hence

$$\cos 36^\circ = \frac{AL}{AX} = \frac{a}{2} \div x = \frac{1}{\sqrt{5} - 1}$$

$$= \frac{\sqrt{5} + 1}{(\sqrt{5} - 1)(\sqrt{5} + 1)} = \frac{\sqrt{5} + 1}{4}.$$

123. To find the trigonometrical functions for an angle of $9^\circ$.

Since $\sin 9^\circ$ and $\cos 9^\circ$ are both positive, the relation (3) of Art. 113 gives

$$\sin 9^\circ + \cos 9^\circ = \sqrt{1 + \sin 18^\circ} = \sqrt{1 + \frac{\sqrt{5} - 1}{4}} = \frac{\sqrt{3} + \sqrt{5}}{2}$$

\ldots \ldots \ldots \ldots (1).$$

Also, since $\cos 9^\circ$ is greater than $\sin 9^\circ$ (Art. 53), the quantity $\sin 9^\circ - \cos 9^\circ$ is negative. Hence the relation (4) of Art. 113 gives

$$\sin 9^\circ - \cos 9^\circ = -\sqrt{1 - \sin 18^\circ} = -\sqrt{1 - \frac{\sqrt{5} - 1}{4}}$$

$$= -\frac{\sqrt{5} - \sqrt{5}}{2} \ldots \ldots \ldots \ldots (2).$$

By adding (1) and (2), we have

$$\sin 9^\circ = \frac{\sqrt{3} + \sqrt{5} - \sqrt{5} - \sqrt{5}}{4},$$

and, by subtracting (2) from (1), we have

$$\cos 9^\circ = \frac{\sqrt{3} + \sqrt{5} + \sqrt{5} - \sqrt{5}}{4}.$$

The remaining functions for $9^\circ$ may now be found.

L. T.
Also, since 81° is the complement of 9°, the values of the functions for 81° may be obtained by the use of Art. 69.

**EXAMPLES. XIX.**

Prove that

1. \[ \sin^2 72\degree - \sin^2 60\degree = \frac{\sqrt{5} - 1}{8}. \]

2. \[ \cos^2 48\degree - \sin^2 12\degree = \frac{\sqrt{5} + 1}{8}. \]

3. \[ \cos 12\degree + \cos 60\degree + \cos 84\degree = \cos 24\degree + \cos 43\degree. \] Verify by a graph.

4. \[ \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5} = \frac{5}{16}. \]

5. \[ \sin \frac{\pi}{10} + \sin \frac{13\pi}{10} = -\frac{1}{2}. \]

6. \[ \sin \frac{\pi}{10} \sin \frac{13\pi}{10} = -\frac{1}{4}. \]

7. \[ \tan 6\degree \tan 42\degree \tan 66\degree \tan 78\degree = 1. \]

8. \[ \cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{5\pi}{15} \cos \frac{6\pi}{15} \cos \frac{7\pi}{15} = \frac{1}{2^7}. \]

9. \[ 16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1. \]

10. **Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.**

11. **In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.**

12. Construct the angle whose cosine is equal to its tangent.

13. Solve the equation

\[ \sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta. \]
CHAPTER IX.

IDENTITIES AND TRIGONOMETRICAL EQUATIONS.

124. The formulae of Arts. 88 and 90 can be used to obtain the trigonometrical ratios of the sum of more than two angles.

For example

\[
\sin (A + B + C) = \sin (A + B) \cos C + \cos (A + B) \sin C
\]

\[
= [\sin A \cos B + \cos A \sin B] \cos C
\]

\[
+ [\cos A \cos B - \sin A \sin B] \times \sin C
\]

\[
= \sin A \cos B \cos C + \cos A \sin B \cos C
\]

\[
+ \cos A \cos B \sin C - \sin A \sin B \sin C.
\]

So

\[
\cos (A + B + C) = \cos (A + B) \cos C - \sin (A + B) \sin C
\]

\[
= (\cos A \cos B - \sin A \sin B) \cos C
\]

\[
- (\sin A \cos B + \cos A \sin B) \sin C
\]

\[
= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C
\]

\[
- \sin A \sin B \cos C,
\]

9—2
Also \[ \tan (A + B + C) = \frac{\tan (A + B) + \tan C}{1 - \tan (A + B) \tan C} \]
\[= \frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C \]
\[= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}. \]

125. The last formula of the previous article is a particular case of a very general theorem which gives the tangent of the sum of any number of angles in terms of the tangents of the angles themselves. The theorem is

\[ \tan (A_1 + A_2 + A_3 + \ldots + A_n) \]
\[= \frac{s_1 - s_3 + s_5 - s_7 + \ldots}{1 - s_2 + s_4 - s_6 + \ldots} \ldots \ldots \ldots \ldots \ldots (1), \]

where

\[ s_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n \]

= the sum of the tangents of the separate angles,

\[ s_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots \]

= the sum of the tangents taken two at a time,

\[ s_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots \]

= the sum of the tangents taken three at a time, and so on.

Assume the relation (1) to hold for \( n \) angles, and add on another angle \( A_{n+1} \).

Then \[ \tan (A_1 + A_2 + \ldots + A_{n+1}) \]
\[= \tan [(A_1 + A_2 + \ldots + A_n) + A_{n+1}] \]
\[= \frac{\tan (A_1 + A_2 + \ldots + A_n) + \tan A_{n+1}}{1 - \tan (A_1 + A_2 + \ldots + A_n) \cdot \tan A_{n+1}} \]
TANGENT OF THE SUM OF ANGLES.

\[ \frac{s_1 - s_3 + s_5 - s_7 + \ldots}{1 - s_2 + s_4 - \ldots} + \tan A_{n+1} \]
\[ = \frac{1}{1 - \frac{s_1 - s_2 + s_4 - \ldots}{1 - s_2 + s_4 - \ldots} \tan A_{n+1}}. \]

Let \( \tan A_1, \tan A_2, \ldots, \tan A_{n+1} \) be respectively called \( t_1, t_2, \ldots, t_{n+1} \).

Then \[ \tan (A_1 + A_2 + \ldots + A_{n+1}) \]
\[ = \frac{(s_1 - s_3 + s_5 \ldots) + t_{n+1} (1 - s_2 + s_4 \ldots)}{(1 - s_2 + s_4 \ldots) - (s_1 - s_3 + s_5 \ldots) t_{n+1}} \]
\[ = \frac{(s_1 + t_{n+1}) - (s_3 + s_2 t_{n+1}) + (s_5 + s_4 t_{n+1}) \ldots}{1 - (s_2 + s_1 t_{n+1}) + (s_4 + s_3 t_{n+1}) - (s_6 + s_5 t_{n+1}) \ldots}. \]

But \( s_1 + t_{n+1} = (t_1 + t_2 + \ldots t_n) + t_{n+1} \)

= the sum of the \((n + 1)\) tangents,

\( s_2 + s_1 t_{n+1} = (t_1 t_2 + t_2 t_3 + \ldots) + (t_1 + t_2 + \ldots + t_n) t_{n+1} \)

= the sum, two at a time, of the \((n + 1)\) tangents.

\( s_3 + s_2 t_{n+1} = (t_1 t_2 t_3 + t_2 t_3 t_4 + \ldots) + (t_1 t_2 + t_2 t_3 + \ldots) t_{n+1} \)

= the sum three at a time of the \((n + 1)\) tangents

and so on.

Hence we see that the same rule holds for \((n + 1)\) angles as for \(n\) angles.

Hence, if the theorem be true for \(n\) angles, it is true for \((n + 1)\) angles.

But, by Arts. 98 and 124, it is true for 2 and 3 angles.

Hence the theorem is true for 4 angles; hence for 5 angles .... Hence it is true universally.

**Cor.** If the angles be all equal, and there be \(n\) of them, and each equal to \(\theta\), then

\[ s_1 = n \cdot \tan \theta; \ s_2 = n C_2 \tan^2 \theta; \ s_3 = n C_3 \tan^3 \theta, \ldots. \]
EX. Write down the value of tan 4\theta.

\[
\tan 4\theta = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{4 \tan \theta - 4C_2 \tan^3 \theta}{1 - 4C_2 \tan^2 \theta + 4C_4 \tan^4 \theta}
\]

\[
= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.
\]

Ex. Prove that \(\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}\).

\(\checkmark\) 126. By a method similar to that of the last article it may be shewn that \(\sin (A_1 + A_2 + \ldots + A_n)\)

\[= \cos A_1 \cos A_2 \ldots \cos A_n (s_1 - s_3 + s_5 - \ldots),\]

and that \(\cos (A_1 + A_2 + \ldots + A_n)\)

\[= \cos A_1 \cos A_2 \ldots \cos A_n (1 - s_2 + s_4 - \ldots),\]

where \(s_1, s_2, s_3, \ldots\) have the same values as in that article.

127. Identities holding between the trigonometrical ratios of the angles of a triangle.

When three angles \(A, B,\) and \(C,\) are such that their sum is 180°, many identical relations are found to hold between their trigonometrical ratios.

The method of proof is best seen from the following examples.

EX. 1. If \(A + B + C = 180^\circ,\) to prove that

\[\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.\]

\[\sin 2A + \sin 2B + \sin 2C = 2 \sin (A + B) \cos (A - B) + 2 \sin C \cos C.\]

Since \(A + B + C = 180^\circ,\)

we have \(A + B = 180^\circ - C,\)

and therefore \(\sin (A + B) = \sin C,\)

and \(\cos (A + B) = - \cos C.\) (Art. 72)
Hence the expression
\[= 2 \sin C \cos (A - B) + 2 \sin C \cos C\]
\[= 2 \sin C [\cos (A - B) + \cos C]\]
\[= 2 \sin C [\cos (A - B) - \cos (A + B)]\]
\[= 2 \sin C \cdot 2 \sin A \sin B\]
\[= 4 \sin A \sin B \sin C.\]

**Ex. 2.** If \(A + B + C = 180^\circ,\)
prove that \(\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.\)

The expression
\[= \cos A + (\cos B - \cos C)\]
\[= 2 \cos^2 \frac{A}{2} - 1 + 2 \sin \frac{B + C}{2} \sin \frac{C - B}{2}.\]

Now \(B + C = 180^\circ - A,\)
so that \(\frac{B + C}{2} = 90^\circ - \frac{A}{2},\)
and therefore \(\sin \frac{B + C}{2} = \cos \frac{A}{2},\)
and \(\cos \frac{B + C}{2} = \sin \frac{A}{2}.\)

Hence the expression
\[= 2 \cos^2 \frac{A}{2} - 1 + 2 \cos \frac{A}{2} \sin \frac{C - B}{2}\]
\[= 2 \cos \frac{A}{2} \left[ \cos \frac{A}{2} + \sin \frac{C - B}{2} \right] - 1\]
\[= 2 \cos \frac{A}{2} \left[ \sin \frac{B + C}{2} + \sin \frac{C - B}{2} \right] - 1\]
\[= 2 \cos \frac{A}{2} \cdot 2 \sin \frac{C}{2} \cos \frac{B}{2} - 1\]
\[= -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.\]

**Ex. 3.** If \(A + B + C = 180^\circ,\)
prove that \(\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.\)

Let \(S = \sin^2 A + \sin^2 B + \sin^2 C,\)
so that \(2S = 2 \sin^2 A + 1 - \cos 2B + 1 - \cos 2C\)
\[= 2 \sin^2 A + 2 - 2 \cos (B + C) \cos (B - C)\]
\[= 2 - 2 \cos^2 A + 2 - 2 \cos (B + C) \cos (B - C).\]
\[\therefore S = 2 + \cos A [\cos (B - C) + \cos (B + C)].\]
since \[ \cos A = \cos \{180^\circ - (B + C)\} = - \cos (B + C). \]

\[ \therefore S = 2 + \cos A \cdot 2 \cos B \cos C. \]

\[ = 2 + 2 \cos A \cos B \cos C. \]

**Ex. 4.** If \( A + B + C = 180^\circ \),

prove that \( \tan A + \tan B + \tan C = \tan A \tan B \tan C \).

By the third formula of Art. 124, we have

\[ \tan (A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B \tan C}. \]

But \( \tan (A + B + C) = \tan 180^\circ = 0. \)

Hence \( 0 = \tan A + \tan B + \tan C - \tan A \tan B \tan C, \)

i.e. \( \tan A + \tan B + \tan C = \tan A \tan B \tan C. \)

This may also be proved independently. For

\[ \tan (A + B) = \tan (180^\circ - C) = - \tan C. \]

\[ \therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = - \tan C. \]

\[ \therefore \tan A + \tan B = - \tan C + \tan A \tan B \tan C, \]

i.e. \( \tan A + \tan B + \tan C = \tan A \tan B \tan C. \)

**Ex. 5.** If \( x + y + z = xyz \), prove that

\[ \frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}. \]

Put \( x = \tan A, y = \tan B, \) and \( z = \tan C, \) so that we have

\[ \tan A + \tan B + \tan C = \tan A \tan B \tan C. \]

\[ \therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = - \tan C, \]

so that \( \tan (A + B) = \tan (x - C). \) \[ \text{[Art. 72.]} \]

Hence \( A + B + C = n\pi + \pi, \)

\[ \therefore \frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C} \]

\[ = \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C, \]

(by a proof similar to that of the last example)

\[ = \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}. \]
EXAMPLES. XX.

If \( A + B + C = 180^\circ \), prove that

1. \( \sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C \).
2. \( \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C \).
3. \( \cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C \).
4. \( \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \).
5. \( \sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \).
6. \( \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).
7. \( \sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C \).
8. \( \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C \).
9. \( \cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C \).
10. \( \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).
11. \( \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} \).
12. \( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{A}{2} = 1 \).
13. \( \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \).
14. \( \cot B \cot C + \cot C \cot A + \cot A \cot B = 1 \).
15. \( \sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B) \)
   \[= 4 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2} \).
16. \( \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1 = 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4} \).
17. \( \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} = 4 \cos \frac{\pi + A}{4} \cos \frac{\pi + B}{4} \cos \frac{\pi - C}{4} \).
18. \( \frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).
19. \[ \sin (B + C - A) + \sin (C + A - B) + \sin (A + B - C) = 4 \sin A \sin B \sin C. \]

If \( A + B + C = 2\pi \) prove that

20. \( \sin (S - A) \sin (S - B) + \sin S \sin (S - C) = \sin A \sin B. \)

21. \[ 4 \sin S \sin (S - A) \sin (S - B) \sin (S - C) = 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C. \]

22. \[ \sin (S - A) + \sin (S - B) + \sin (S - C) - \sin S = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \]

23. \[ \cos^2 S + \cos^2 (S - A) + \cos^2 (S - B) + \cos^2 (S - C) = 2 + 2 \cos A \cos B \cos C. \]

24. \[ \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1 + 4 \cos S \cos (S - A) \cos (S - B) \cos (S - C). \]

25. If \( a + b + \gamma + \delta = 2\pi \), prove that

(1) \[ \cos a + \cos b + \cos \gamma + \cos \delta + 4 \cos \frac{a + b}{2} \cos \frac{a + \gamma}{2} \cos \frac{a + \delta}{2} = 0, \]

(2) \[ \sin a - \sin b + \sin \gamma - \sin \delta + 4 \cos \frac{a + b}{2} \sin \frac{a + \gamma}{2} \cos \frac{a + \delta}{2} = 0, \]

and (3) \[ \tan a + \tan b + \tan \gamma + \tan \delta = \tan a + \tan b + \tan \gamma + \tan \delta. \]

26. If the sum of four angles be \( 180^\circ \), prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

27. Prove that \( \sin 2a + \sin 2b + \sin 2\gamma \)

\[ = 2 (\sin a + \sin b + \sin \gamma) (1 + \cos a + \cos b + \cos \gamma), \]

if \( a + b + \gamma = 0. \)

28. Verify that

\[ \sin^3 a \sin (b - c) + \sin^3 b \sin (c - a) + \sin^3 c \sin (a - b) \]

\[ + \sin (a + b + c) \sin (b + c) \sin (c - a) \sin (a - b) = 0. \]

If \( A, B, C, \) and \( D \) be any angles prove that

29. \[ \sin A \sin B \sin (A - B) + \sin B \sin C \sin (B - C) \]

\[ + \sin C \sin A \sin (C - A) + \sin (A - B) \sin (B - C) \sin (C - A) = 0. \]
30. \[ \sin (A - B) \cos (A + B) + \sin (B - C) \cos (B + C) + \sin (C - D) \cos (C + D) + \sin (D - A) \cos (D + A) = 0. \]

31. \[ \sin (A + B - 2C) \cos B - \sin (A + C - 2B) \cos C = \sin (B - C) \left\{ \cos (B + C - A) + \cos (C + A - B) + \cos (A + B - C) \right\}. \]

32. \[ \sin (A + B + C + D) + \sin (A + B - C - D) + \sin (A + B - C + D) + \sin (A + B + C - D) = 4 \sin (A + B) \cos C \cos D. \]

33. If any theorem be true for values of \( A, B, \) and \( C \) such that \( A + B + C = 180^\circ, \)

prove that the theorem is still true if we substitute for \( A, B, \) and \( C \)
respectively the quantities

\[ (1) \quad 90^\circ - \frac{A}{2}, \quad 90^\circ - \frac{B}{2}, \quad \text{and} \quad 90^\circ - \frac{C}{2}, \]

or

\[ (2) \quad 180^\circ - 2A, \quad 180^\circ - 2B, \quad \text{and} \quad 180^\circ - 2C. \]

Hence deduce Ex. 16 from Ex. 6, and Ex. 17 from Ex. 5.

If \( x + y + z = xyz \) prove that

\[ \frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}. \]

and 35. \[ x (1 - y^2) (1 - z^2) + y (1 - z^2) (1 - x^2) + z (1 - x^2) (1 - y^2) = 4xyz. \]

128. The Addition and Subtraction Theorems may be used to solve some kinds of trigonometrical equations.

**Ex.** Solve the equation

\[ \sin x + \sin 5x = \sin 3x. \]

By the formulae of Art. 94, the equation is

\[ 2 \sin 3x \cos 2x = \sin 3x. \]

\[ \therefore \quad \sin 3x = 0, \quad \text{or} \quad 2 \cos 2x = 1. \]

If \( \sin 3x = 0, \) then \( 3x = n\pi. \)

If \( \cos 2x = \frac{1}{2}, \) then \( 2x = 2n\pi \pm \frac{\pi}{3}. \)

Hence \( x = \frac{n\pi}{3}, \) or \( n\pi \pm \frac{\pi}{6}. \)
129. **To solve an equation of the form**

\[ a \cos \theta + b \sin \theta = c. \]

Divide both sides of the equation by \( \sqrt{a^2 + b^2} \), so that it may be written

\[
\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta = \frac{c}{\sqrt{a^2 + b^2}}.
\]

Find from the table of tangents the angle whose tangent is \( \frac{b}{a} \) and call it \( \alpha \).

Then \( \tan \alpha = \frac{b}{a} \), so that

\[ \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \text{ and } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}. \]

The equation can then be written

\[ \cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{c}{\sqrt{a^2 + b^2}}, \]

i.e.

\[ \cos (\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}. \]

Next find from the tables, or otherwise, the angle \( \beta \)

whose cosine is

\[ \frac{c}{\sqrt{a^2 + b^2}}, \]

so that

\[ \cos \beta = \frac{c}{\sqrt{a^2 + b^2}}, \]

[N.B. This can only be done when \( c < \sqrt{a^2 + b^2} \).]

The equation is then \( \cos (\theta - \alpha) = \cos \beta \).

The solution of this is \( \theta - \alpha = 2n\pi \pm \beta \), so that

\[ \theta = 2n\pi + \alpha \pm \beta, \]

where \( n \) is any integer.
Angles, such as $\alpha$ and $\beta$, which are introduced into trigonometrical work to facilitate computation are called **Subsidiary Angles**.

130. The above solution may be illustrated graphically as follows;

Measure $OM$ along the initial line equal to $a$, and $MP$ perpendicular to it, and equal to $b$. The angle $MOP$ is then the angle whose tangent is $\frac{b}{a}$, i.e. $\alpha$.

With centre $O$ and radius $OP$, i.e. $\sqrt{a^2+b^2}$, describe a circle, and measure $ON$ along the initial line equal to $c$.

Draw $QNQ'$ perpendicular to $ON$ to meet the circle in $Q$ and $Q'$; the angles $NOQ$ and $Q'ON$ are therefore each equal to $\beta$.

The angle $QOP$ is therefore $\alpha - \beta$ and $Q'OP$ is $\alpha + \beta$.

Hence the solutions of the equation are respectively

$$2n\pi + QOP \text{ and } 2n\pi + Q'OP.$$  

The construction clearly fails if $c > \sqrt{a^2+b^2}$, for then the point $N$ would fall outside the circle.

131. As a numerical example let us solve the equation

$$5 \cos \theta - 2 \sin \theta = 2\sqrt{29},$$
given that $\tan 21^\circ 48' = \frac{2}{5}$.

Dividing both sides of the equation by $\sqrt{5^2+2^2}$, i.e. $\sqrt{29}$,

we have

$$\frac{5}{\sqrt{29}} \cos \theta - \frac{2}{\sqrt{29}} \sin \theta = \frac{2}{\sqrt{29}}.$$
Hence
\[ \cos \theta \cos 21^\circ 48' - \sin \theta \sin 21^\circ 48' \]
\[ = \sin 21^\circ 48' = \sin (90^\circ - 68^\circ 12') \]
\[ = \cos 68^\circ 12'. \]
\[ \therefore \cos (\theta + 21^\circ 48') = \cos 68^\circ 12'. \]

Hence
\[ \theta + 21^\circ 48' = 2n \times 180^\circ \pm 68^\circ 12'. \tag{Art. 83} \]
\[ \therefore \theta = 2n \times 180^\circ - 21^\circ 48' \pm 68^\circ 12' \]
\[ = 2n \times 180^\circ - 90^\circ, \text{ or } 2n \times 180^\circ + 46^\circ 24', \]
where \( n \) is any integer.

\textbf{Aliter.} The equation of Art. 129 may be solved in another way.

For let \( t = \tan \frac{\theta}{2} \),

so that
\[ \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \text{tan}^2 \frac{\theta}{2}} = \frac{2t}{1 + t^2}, \]

and
\[ \cos \theta = \frac{1 - \text{tan}^2 \frac{\theta}{2}}{1 + \text{tan}^2 \frac{\theta}{2}} = \frac{1 - t^2}{1 + t^2}. \tag{Art. 109.} \]

The equation then becomes
\[ a \frac{1 - t^2}{1 + t^2} + b \frac{2t}{1 + t^2} = c, \]
so that
\[ t^2 (c + a) - 2bt + c - a = 0. \]

This is a quadratic equation giving two values for \( t \) and hence two values for \( \tan \frac{\theta}{2} \).

Thus, the example of this article gives
\[ 7t^2 + 4t - 3 = 0, \]
so that
\[ t = -1 \text{ or } \frac{3}{7}, \]
\[ = \tan (-45^\circ) \text{ or } \tan 23^\circ 12' \text{ (from the tables)}. \]

Hence
\[ \frac{\theta}{2} = n \cdot 180^\circ - 45^\circ, \text{ or } n \cdot 180^\circ + 23^\circ 12', \]

i.e.
\[ \theta = n \cdot 360^\circ - 90^\circ, \text{ or } n \cdot 360^\circ + 46^\circ 24'. \]
EXAMPLES. XXI.

Solve the equations

1. \( \sin \theta + \sin 7\theta = \sin 4\theta. \)
2. \( \cos \theta + \cos 7\theta = \cos 4\theta. \)
3. \( \cos \theta + \cos 3\theta = 2 \cos 2\theta. \)
4. \( \sin 4\theta - \sin 2\theta = \cos 3\theta. \)
5. \( \cos \theta - \sin 3\theta = \cos 2\theta. \)
6. \( \sin 7\theta = \sin \theta + \sin 3\theta. \)
7. \( \cos \theta + \cos 2\theta + \cos 3\theta = 0. \)
8. \( \sin \theta + \sin 3\theta + \sin 5\theta = 0. \)
9. \( \sin 2\theta - \cos 2\theta - \sin \theta + \cos \theta = 0. \)
10. \( \sin (3\theta + a) + \sin (3\theta - a) + \sin (a - \theta) - \sin (a + \theta) = \cos a. \)
11. \( \cos (3\theta + a) \cos (3\theta - a) + \cos (5\theta + a) \cos (5\theta - a) = \cos 2a. \)
12. \( \cos n\theta = \cos (n - 2) \theta + \sin \theta. \)
13. \( \sin \frac{n+1}{2} \theta = \sin \frac{n-1}{2} \theta + \sin \theta. \)
14. \( \sin m\theta + \sin n\theta = 0. \)
15. \( \cos m\theta + \cos n\theta = 0. \)
16. \( \sin^2 n\theta - \sin^2 (n - 1) \theta = \sin \theta. \)
17. \( \sin 3\theta + \cos 2\theta = 0. \)
18. \( \sqrt{3} \cos \theta + \sin \theta = \sqrt{2}. \)
19. \( \sin \theta + \cos \theta = \sqrt{2}. \)
20. \( \sqrt{3} \sin \theta - \cos \theta = \sqrt{2}. \)
21. \( \sin x + \cos x = \sqrt{2} \cos \theta. \)
22. \( 5 \sin \theta + 2 \cos \theta = 5 \) (given \( \tan 21^\circ 18' = 0.4 \).)
23. \( 6 \cos x + 8 \sin x = 9 \) (given \( \tan 53^\circ 18' = 1.4 \) and \( \cos 25^\circ 50' = 0.9 \).)
24. \( 1 + \sin^2 \theta = 3 \sin \theta \cos \theta \) (given \( \tan 71^\circ 34' = 3 \).)
25. \( \cosec \theta = \cot \theta + \sqrt{3}. \)
26. \( \cosec x = 1 + \cot x. \)
27. \( (2 + \sqrt{3}) \cos \theta = 1 - \sin \theta. \)
28. \( \tan \theta + \sec \theta = \sqrt{3}. \)
29. \( \cos 2\theta = \cos^2 \theta. \)
30. \( 4 \cos \theta - 3 \sec \theta = \tan \theta. \)
31. \( \cos 2\theta + 3 \cos \theta = 0. \)
32. \( \cos 3\theta + 2 \cos \theta = 0. \)
33. \( \cos 2\theta = (\sqrt{2} + 1) \left( \cos \theta - \frac{1}{\sqrt{2}} \right). \)
34. \( \cot \theta - \tan \theta = 2. \)
35. \( 4 \cot 2\theta = \cot^2 \theta - \tan^2 \theta. \)
36. \( 3 \tan (\theta - 15^\circ) = \tan (\theta + 15^\circ). \)
37. \( \tan \theta + \tan 2\theta + \tan 3\theta = 0. \)
38. \( \tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}. \)
39. \( \sin 3a = 4 \sin a \sin (x + a) \sin (x - a). \)
40. Prove that the equation \( x^3 - 2x + 1 = 0 \) is satisfied by putting for \( x \) either of the values \( \sqrt{2} \sin 45^\circ, 2 \sin 18^\circ, \) and \( 2 \sin 234^\circ. \)
41. If \( \sin (\pi \cos \theta) = \cos (\pi \sin \theta) \), prove that \( \cos \left( \theta \pm \frac{\pi}{4} \right) = \frac{1}{2\sqrt{2}} \).

42. If \( \sin (\pi \cot \theta) = \cos (\pi \tan \theta) \), prove that either cosec \( 2\theta \) or cot \( 2\theta \) is equal to \( n + \frac{1}{2} \) where \( n \) is a positive or negative integer.

132. Ex. To trace the changes in the expression \( \sin x + \cos x \) as \( x \) increases from 0 to \( 2\pi \).

We have \( \sin x + \cos x = \sqrt{2} \left[ \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right] \)

\[ = \sqrt{2} \left[ \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} \right] = \sqrt{2} \sin \left( x + \frac{\pi}{4} \right). \]

We thus have the following table of values:

| \( x \) | \( 0 \) | \( \frac{\pi}{4} \) | \( \frac{3\pi}{4} \) | \( \frac{5\pi}{4} \) | \( \frac{7\pi}{4} \) | \( 2\pi \) |
| \hline
| \( x + \frac{\pi}{4} \) | \( \frac{\pi}{4} \) | \( \frac{\pi}{2} \) | \( \pi \) | \( \frac{3\pi}{2} \) | \( 2\pi \) | \( \frac{9\pi}{4} \) |
| \hline
| \( \sin \left( x + \frac{\pi}{4} \right) \) | \( \frac{1}{\sqrt{2}} \) | 1 | 0 | -1 | 0 | \( \frac{1}{\sqrt{2}} \) |
| \hline
| \( \sqrt{2} \sin \left( x + \frac{\pi}{4} \right) \) | 1 | \( \sqrt{2} \) | 0 | -\( \sqrt{2} \) | 0 | 1 |

As in Art. 62, the graph is as in the following figure.

133. Ex. To trace the changes in the sign and magnitude of \( a \cos \theta + b \sin \theta \), and to find the greatest value of the expression.

We have

\[ a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \left[ \frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta \right] \]
Let $\alpha$ be the smallest positive angle such that
\[
\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \text{ and } \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}.
\]

The expression therefore
\[
= \sqrt{a^2 + b^2} \left[ \cos \theta \cos \alpha + \sin \theta \sin \alpha \right] = \sqrt{a^2 + b^2} \cos (\theta - \alpha).
\]

As $\theta$ changes from $\alpha$ to $2\pi + \alpha$, the angle $\theta - \alpha$ changes from 0 to $2\pi$, and hence the changes in the sign and magnitude of the expression are easily obtained.

Since the greatest value of the quantity $\cos (\theta - \alpha)$ is unity, i.e. when $\theta$ equals $\alpha$, the greatest value of the expression is $\sqrt{a^2 + b^2}$.

Also the value of $\theta$ which gives this greatest value is such that its cosine is $\frac{a}{\sqrt{a^2 + b^2}}$.

**EXAMPLES. XXII.**

As $\theta$ increases from 0 to $2\pi$, trace the changes in the sign and magnitude of the following expressions, and plot their graphs.

1. $\sin \theta - \cos \theta$.
2. $\sin \theta + \sqrt{3} \cos \theta$.
   \[
   \left[ \text{N.B. } \sin \theta + \sqrt{3} \cos \theta = 2 \left( \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta \right) = 2 \sin \left( \theta + \frac{\pi}{3} \right) \right].
   \]
3. $\sin \theta - \sqrt{3} \cos \theta$.
4. $\cos^2 \theta - \sin^2 \theta$.
5. $\sin \theta \cos \theta$.
6. $\sin 3\theta$.
7. $\tan 3\theta$.
8. $\sec 4\theta$.
9. $\frac{\sin \theta + \sin 2\theta}{\cos \theta + \cos 2\theta}$.
10. $\sin (\pi \sin \theta)$.
11. $\cos (\pi \sin \theta)$.
12. Trace the changes in the sign and magnitude of $\frac{\sin 3\theta}{\cos 2\theta}$ as the angle increases from 0 to $90^\circ$.

L. T. 10
CHAPTER X.

LOGARITHMS.

134. SUPPOSING that we know that

\[ 10^{2.4031205} = 253, \quad 10^{2.0095944} = 407, \]

and

\[ 10^{5.0127149} = 102971, \]

we can shew that \( 253 \times 407 = 102971 \) without performing the operation of multiplication. For

\[ 253 \times 407 = 10^{2.4031205} \times 10^{2.0095944} = 10^{2.4031205+2.0095944} = 10^{5.0127149} = 102971. \]

Here it will be noticed that the process of multiplication has been replaced by the simpler process of addition.

Again, supposing that we know that

\[ 10^{4.9004055} = 79507, \]

and that

\[ 10^{1.6334683} = 43, \]

we can easily shew that the cube root of 79507 is 43.

For \[ \sqrt[3]{79507} = [79507]^{\frac{1}{3}} = (10^{4.9004055})^{\frac{1}{3}} = 10^{\frac{1}{3} \times 4.9004055} = 10^{1.6334683} = 43. \]

Here it will be noticed that the difficult process of extracting the cube root has been replaced by the simpler process of division.
135. **Logarithm.** Def. *If a be any number, and x and N two other numbers such that \( a^x = N \), then x is called the logarithm of N to the base a and is written \( \log_a N \).*

The logarithm of a number to a given base is therefore the index of the power to which the base must be raised that it may be equal to the given number.

*Exs.* Since \( 10^2 = 100 \), therefore \( 2 = \log_{10} 100 \).
Since \( 10^4 = 100000 \), therefore \( 5 = \log_{10} 100000 \).
Since \( 2^4 = 16 \), therefore \( 4 = \log_2 16 \).
Since \( 8^{\frac{3}{2}} = [8^{\frac{1}{2}}]^2 = 4 \), therefore \( \frac{2}{3} = \log_8 4 \).
Since \( 9^{-\frac{2}{3}} = \frac{1}{9^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{9^2}} = \frac{1}{27} \), therefore

\[
-\frac{2}{3} = \log_9 \left( \frac{1}{27} \right).
\]

N.B. Since \( a^0 = 1 \) always, the logarithm of unity to any base is always zero.

136. In Algebra, if \( m \) and \( n \) be any real quantities whatever, the following laws, known as the laws of indices, are found to be true:

\[
\begin{align*}
\text{(i)} & \quad a^m \times a^n = a^{m+n}, \\
\text{(ii)} & \quad a^m \div a^n = a^{m-n}, \\
\text{and} & \quad (a^m)^n = a^{mn}.
\end{align*}
\]

Corresponding to these we have three fundamental laws of logarithms, viz.

\[
\begin{align*}
\text{(i)} & \quad \log_a (mn) = \log_a m + \log_a n, \\
\text{(ii)} & \quad \log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n, \\
\text{and} & \quad \log_a m^n = n \log_a m.
\end{align*}
\]

The proofs of these laws are given in the following articles.

137. The logarithm of the product of two quantities is

\[10-2\]
equal to the sum of the logarithms of the quantities to the same base, i.e.

$$\log_a (mn) = \log_a m + \log_a n.$$  

Let $x = \log_a m$, so that $a^x = m$,
and $y = \log_a n$, so that $a^y = n$.
Then

$$mn = a^x \times a^y = a^{x+y}.$$  

$\therefore \log_a mn = x + y$ (Art. 135, Def.)

$$= \log_a m + \log_a n.$$

138. The logarithm of the quotient of two quantities is equal to the difference of their logarithms, i.e.

$$\log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n.$$  

Let $x = \log_a m$, so that $a^x = m$, (Art. 135, Def.)
and $y = \log_a n$, so that $a^y = n$.
Then

$$\frac{m}{n} = a^x \div a^y = a^{x-y}.$$  

$\therefore \log_a \left( \frac{m}{n} \right) = x - y$ (Art. 135, Def.)

$$= \log_a m - \log_a n.$$

139. The logarithm of a quantity raised to any power is equal to the logarithm of the quantity multiplied by the index of the power, i.e.

$$\log_a (m^n) = n \log_a m.$$  

Let $x = \log_a m$, so that $a^x = m$. Then

$$m^n = (a^x)^n = a^{nx}.$$  

$\therefore \log_a (m^n) = nx$ (Art. 135, Def.)

$$= n \log_a m.$$

**Exs.**

$$\log 48 = \log (2^4 \times 3) = \log 2^4 + \log 3 = 4 \log 2 + \log 3;$$

$$\log \frac{63}{484} = \log 7 \times 3^2 = \log 7 + \log 3^2 - \log 2^2 - \log 11^2$$

$$= \log 7 + 2 \log 3 - 2 \log 2 - 2 \log 11;$$

$$\log \sqrt[3]{13} = \log 13^{\frac{1}{3}} = \frac{1}{3} \log 13.$$
140. **Common system of logarithms.** In the system of logarithms which we practically use the base is always 10, so that, if no base be expressed, the base 10 is always understood. The advantage of using 10 as the base is seen in the three following articles.

141. **Characteristic and Mantissa.** Def. If the logarithm of any number be partly integral and partly fractional, the integral portion of the logarithm is called its characteristic and the decimal portion is called its mantissa.

Thus, supposing that $\log 795 = 2.9003671$, the number 2 is the characteristic and $0.9003671$ is the mantissa.

**Negative characteristics.** Suppose we know that $\log 2 = 0.30103$.

Then, by Art. 138,

$$\log \frac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = -0.30103,$$

so that $\log \frac{1}{2}$ is negative.

Now it is found convenient, as will be seen in Art. 143, that the mantissae of all logarithms should be kept positive. We therefore instead of $-0.30103$ write $[1 - 0.69897]$, so that

$$\log \frac{1}{2} = -(1 - 0.69897) = -1 + 0.69897.$$

For shortness this latter expression is written $1.69897$.

The horizontal line over the 1 denotes that the integral part is negative; the decimal part however is positive.

As another example, $3.4771213$ stands for

$$-3 + 0.4771213.$$

142. **The characteristic of the logarithm of any number can always be determined by inspection.**
(i) Let the number be greater than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^1 = 10$, therefore $\log 10 = 1$;

since $10^2 = 100$, therefore $\log 100 = 2$,

and so on.

Hence the logarithm of any number lying between 1 and 10 must lie between 0 and 1, that is, it will be a decimal fraction and therefore have 0 as its characteristic.

So the logarithm of any number between 10 and 100 must lie between 1 and 2, i.e. it will have a characteristic equal to 1.

Similarly, the logarithm of any number between 100 and 1000 must lie between 2 and 3, i.e. it will have a characteristic equal to 2.

So, if the number lie between 1000 and 10000, the characteristic will be 3.

Generally, the characteristic of the logarithm of any number will be one less than the number of digits in its integral part.

**Exs.** The number 296.3457 has 3 figures in its integral part, and therefore the characteristic of its logarithm is 2.

The characteristic of the logarithm of 29634.57 will be $5 - 1$, i.e. 4.

(ii) Let the number be less than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^{-1} = \frac{1}{10} = .1$, therefore $\log .1 = -1$;

since $10^{-2} = \frac{1}{10^2} = .01$, therefore $\log .01 = -2$;

since $10^{-3} = \frac{1}{10^3} = .001$, therefore $\log .001 = -3$;

and so on.
CHARACTERISTIC OF ANY LOGARITHM. 151

The logarithm of any number between 1 and 1 therefore lies between 0 and −1, and so is equal to −1 + some decimal, i.e. its characteristic is 1.

So the logarithm of any number between 0.1 and 0.01 lies between −1 and −2, and hence it is equal to −2 + some decimal, i.e. its characteristic is 2.

Similarly, the logarithm of any number between 0.01 and 0.001 lies between −2 and −3, i.e. its characteristic is 3.

Generally, the characteristic of the logarithm of any decimal fraction will be negative and numerically will be greater by unity than the number of cyphers following the decimal point.

For any fraction between 1 and 0.1 (e.g. 0.05) has no cypher following the decimal point and we have seen that its characteristic is 1.

Any fraction between 0.1 and 0.01 (e.g. 0.007) has one cypher following the decimal point and we have seen that its characteristic is 2.

Any fraction between 0.01 and 0.001 (e.g. 0.003) has two cyphers following the decimal point and we have seen that its characteristic is 3.

Similarly for any fraction.

Eqs. The characteristic of the logarithm of the number 0.00835 is 3.
The characteristic of the logarithm of the number 0.0000053 is 6.
The characteristic of the logarithm of the number 0.34567 is 1.

143. The mantissae of the logarithm of all numbers, consisting of the same digits, are the same.
This will be made clear by an example.
Suppose we are given that

\[ \log 66818 = 4.8248935. \]
Then
\[
\log 668.18 = \log \frac{66818}{100} = \log 66818 - \log 100 \quad \text{(Art. 138)}
\]
\[
= 4.8248935 - 2 = 2.8248935 ;
\]
\[
\log .66818 = \log \frac{66818}{100000} = \log 66818 - \log 100000
\qquad \text{(Art. 138)}
\]
\[
= 4.8248935 - 5 = 1.8248935.
\]

So
\[
\log .00066818 = \log \frac{66818}{10^8} = \log 66818 - \log 10^8
\]
\[
= 4.8248935 - 8 = 4.8248935.
\]

Now the numbers 66818, 668.18, .66818, and .00066818 consist of the same significant figures, and only differ in the position of the decimal point. We observe that their logarithms have the same decimal portion, i.e. the same mantissa, and they only differ in the characteristic.

The value of this characteristic is in each case determined by the rule of the previous article.

It will be noted that the mantissa of a logarithm is always positive.

144. **Tables of logarithms.** The logarithms of all numbers from 1 to 108000 are given in Chambers’ Tables of Logarithms. Their values are there given correct to seven places of decimals.

The student should have access to a copy of the above table of logarithms or to some other suitable table. It will be required for many examples in the course of the next few chapters.

On the opposite page is a specimen page selected from Chambers’ Tables. It gives the mantissae of the logarithms of all whole numbers from 52500 to 53000.
<table>
<thead>
<tr>
<th>No.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5250</td>
<td>720</td>
<td>1593</td>
<td>1676</td>
<td>1758</td>
<td>1841</td>
<td>1924</td>
<td>2007</td>
<td>2089</td>
<td>2172</td>
<td>2255</td>
<td>2337</td>
</tr>
<tr>
<td>5261</td>
<td>721</td>
<td>0683</td>
<td>0766</td>
<td>0848</td>
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<td>1013</td>
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<td>1178</td>
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<td>1343</td>
<td>1426</td>
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<td>8930</td>
<td>9013</td>
<td>9093</td>
<td>9177</td>
<td>9260</td>
<td>9342</td>
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<td>9507</td>
<td>9589</td>
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<td>7214</td>
<td>7266</td>
<td>7318</td>
<td>7370</td>
<td>7422</td>
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<td>7578</td>
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<td>3005</td>
<td>3087</td>
<td>3169</td>
<td>3251</td>
<td>3333</td>
<td>3415</td>
<td>3497</td>
<td></td>
</tr>
</tbody>
</table>

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#### Notes
- The table above contains numerical data with columns labeled from 0 to 9. Each row represents a sequence of numbers, with the first row being 5250 and the last row being 5300. The final row is marked as Diff., indicating a potential difference or calculation column.
- The data appears to be a continuous sequence, possibly representing a series or a set of related measurements.
145. To obtain the logarithm of any such number, such as 52687, we proceed as follows. Run the eye down the extreme left-hand column until it arrives at the number 5268. Then look horizontally until the eye sees the figures 7035 which are vertically beneath the number 7 at the top of the page. The number corresponding to 52687 is therefore 7217035. But this last number consists only of the digits of the mantissa, so that the mantissa required is .7217035. But the characteristic for 52687 is 4.

Hence \( \log 52687 = 4.7217035 \).

So \( \log .52687 = 1.7217035 \),

and \( \log .00052687 = 4.7217035 \).

If, again, the logarithm of 52725 be required, the student will find (on running his eye vertically down the extreme left-hand column as far as 5272 and then horizontally along the row until he comes to the column under the digit 5) the number 0166. The bar which is placed over these digits denotes that to them must be prefixed not 721 but 722. Hence the mantissa corresponding to the number 52725 is .7220166.

Also the characteristic of the logarithm of the number 52725 is 4.

Hence \( \log 52725 = 4.7220166 \).

So \( \log .052725 = 2.7220166 \).

We shall now work a few numerical examples to shew the efficiency of the application of logarithms for purposes of calculation.

146. Ex. 1. Find the value of \( \sqrt[5]{23.4} \).

Let \( x = \sqrt[5]{23.4} = (23.4)^{\frac{1}{5}} \),

so that \( \log x = \frac{1}{5} \log (23.4) \), by Art. 139.
EXAMPLES OF LOGARITHMS.

In the table of logarithms we find, opposite the number 234, the logarithm 3692159.

Hence \[ \log 23.4 = 1.3692159. \]

Therefore \[ \log x = \frac{1}{5} [1.3692159] = 0.2738432. \]

Again, in the table of logarithms we find, corresponding to the logarithm 2738432, the number 187864, so that
\[ \log 1.87864 = 0.2738432. \]

\[ \therefore x = 1.87864. \]

**Ex. 2.** Find the value of
\[ \frac{(6.45)^3 \times \sqrt[3]{0.00034}}{(9.37)^2 \times \sqrt{8.93}}. \]

Let \( x \) be the required value so that, by Arts. 138 and 139,
\[ \log x = \log (6.45)^3 + \log (0.00034)^\frac{1}{3} - \log (9.37)^2 - \log \sqrt{8.93} \]
\[ = 3 \log (6.45) + \frac{1}{3} \log (0.00034) - 2 \log (9.37) - \frac{1}{4} \log 8.93. \]

Now in the table of logarithms we find

<p>| | | | | |</p>
<table>
<thead>
<tr>
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<td></td>
<td>937</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>893</td>
<td></td>
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</tr>
</tbody>
</table>

opposite the number 645 the logarithm 8095597,
\[ \log x = 3 \times 0.8095597 + \frac{1}{3} \log (6.45) \]
\[ = 2 \times 0.9717396 - \frac{1}{4} \times 0.9508515. \]

Hence
\[ \frac{1}{8} (4.5314789) = \frac{1}{3} [6 + 2.5314789] \]
\[ = 2 + 0.8438263. \]

But \[ \log x = 2.4286791 + [2 + 0.8438263] - 1.9434792 - 2377129 \]
\[ = 3 \times 2725054 - 4 \times 1811921 \]
\[ = 1 + 4 \times 2725054 - 4 \times 1811921. \]
\[ = 1.0913133. \]
In the table of logarithms we find, opposite the number 12340, the logarithm 0913152, so that

\[ \log 12340 = 1.0913152. \]

Hence \( \log x = \log 12340 \) nearly,
and therefore \( x = 12340 \) nearly.

When the logarithm of any number does not quite agree with any logarithm in the tables, but lies between two consecutive logarithms, it will be shewn in the next chapter how the number may be accurately found.

**Ex. 3.** Having given \( \log 2 = 0.30103 \), find the number of digits in \( 2^{37} \) and the position of the first significant figure in \( 2^{-37} \).

We have \[
\log 2^{37} = 37 \times \log 2 = 67 \times 0.30103 = 20.16001.
\]

Since the characteristic of the logarithm of \( 2^{37} \) is 20, it follows, by Art. 142, that in \( 2^{37} \) there are 21 digits.

Again, \[
\log 2^{-37} = -37 \log 2 = -37 \times 0.30103 = -11.13811 = 12.86189.
\]

Hence, by Art. 142, in \( 2^{-37} \) there are 11 cyphers following the decimal point, i.e. the first significant figure is in the twelfth place of decimals.

**Ex. 4.** Given \( \log 3 = 1771213, \log 7 = 8150980, \) and \( \log 11 = 1.0113297 \), solve the equation \( 3^x \times 7^{2x+1} = 11^{x+5} \).

Taking logarithms of both sides we have

\[
\log 3^x + \log 7^{2x+1} = \log 11^{x+5}.
\]

\[\therefore x \log 3 + (2x + 1) \log 7 = (x + 5) \log 11.\]

\[\therefore x[\log 3 + 2 \log 7 - \log 11] = 5 \log 11 - \log 7.\]

\[\therefore x = \frac{5 \log 11 - \log 7}{\log 3 + 2 \log 7 - \log 11}\]

\[= \frac{5 \times 2069635 - 8150980}{1.771213 + 1.6901960 - 1.0113297} = 3.87555 = 3.87555 = 3.87....\]
147. To prove that
\[
\log_a m = \log_b m \times \log_a b.
\]
Let \( \log_a m = x, \) so that \( a^x = m. \)
Also let \( \log_b m = y, \) so that \( b^y = m. \)
\[
\therefore \ a^x = b^y.
\]
Hence \( \log_a (a^x) = \log_a (b^y). \)
\[
\therefore \ x = y \log_a b. \quad (\text{Art. 139.})
\]
Hence \( \log_a m = \log_b m \times \log_a b. \)

By the theorem of the foregoing article we can from the logarithm of any number to a base \( b \) find its logarithm to any other base \( a. \) It is found convenient, as will appear in a subsequent chapter, not to calculate the logarithms to base 10 directly, but to calculate them first to another base and then to transform them by this theorem.

EXAMPLES. XXIII.

1. Given \( \log 4 = 0.60206 \) and \( \log 3 = 0.4771213, \) find the logarithms of \( 0.8, \ 0.003, \ 0.0108, \) and \( (0.00018)^{\frac{1}{2}}. \)

2. Given \( \log 11 = 1.0113927 \) and \( \log 13 = 1.1139434, \) find the values of (1) \( \log 1.43, \) (2) \( \log 133.1, \) (3) \( \log \sqrt[3]{143}, \) and (4) \( \log \sqrt[3]{0.0169}. \)

3. What are the characteristics of the logarithms of \( 243.7, \ 0.0153, \ 2.8713, \ 0.0057, \ 0.023, \ \sqrt[4]{24615}, \) and \( (24589)^{\frac{1}{2}}? \)

4. Find the 5th root of \( 0.003, \) having given \( \log 3 = 0.4771213 \) and \( \log 312936 = 5.4954243. \)

5. Find the value of (1) \( 7^{\frac{1}{4}}, \) (2) \( 84^{\frac{1}{2}}, \) and (3) \( 0.021^{\frac{1}{3}}, \) having given \( \log 2 = 0.30103, \) \( \log 3 = 0.4771213, \)
\[
\log 7 = 0.8450980, \) \( \log 132057 = 5.1207283, \)
\[
\log 588453 = 5.7697117, \text{ and } \log 461791 = 5.6644433.
\]
6. Having given \( \log 3 = .4771213 \), find the number of digits in

(1) \( 3^{43} \), (2) \( 3^{27} \), and (3) \( 3^{62} \),

and the position of the first significant figure in

(4) \( 3^{-13} \), (5) \( 3^{-43} \), and (6) \( 3^{-65} \).

7. Given \( \log 2 = .30103 \), \( \log 3 = .4771213 \), and \( \log 7 = .8450980 \), solve the equations

(1) \( 2^x \cdot 3^{x+4} = 7^x \),

(2) \( 2^{2x+1} \cdot 3^{3x+2} = 7^4x \),

(3) \( 7^{2x} \div 2^{x-4} = 3^{3x-7} \).

and

(4) \[ \begin{align*}
7^x + v \times 3^{2x+v} &= 9 \\
3^{x-v} + 2^{x-2v} &= 3^x
\end{align*} \]

8. From the tables find the seventh root of \( 0.00020751 \).

Making use of the tables, find the approximate values of

9. \( \sqrt[7]{645.3} \).

10. \( \sqrt[8]{2357} \).

11. \( \sqrt[5]{7} \times \sqrt[7]{9} \).

12. \( \sqrt[3]{7.2 \times 8.3} \).

13. \( \sqrt[7]{8.5 \times 11\frac{1}{2}} \).

14. \( \log x \).

15. \( \log \sin x \).

16. \( \log \cos x \).

17. \( \log \tan x \).

18. \( \log \sec x \).

19. \( \log \cot x \).
CHAPTER XI.

TABLES OF LOGARITHMS AND TRIGONOMETRICAL RATIOS.
PRINCIPLE OF PROPORTIONAL PARTS.

148. We have pointed out that the logarithms of all numbers from 1 to 108000 may be found in Chambers' Mathematical Tables, so that, for example, the logarithms of 74583 and 74584 may be obtained directly therefrom.

Suppose however we wanted the logarithm of a number lying between these two, e.g. the number 74583.3.

To obtain the logarithm of this number we use the Principle of Proportional Parts which states that the increase in the logarithm of a number is proportional to the increase in the number itself.

Thus from the tables we find

\[ \log 74583 = 4.8726398 \ldots \ldots \ldots (1), \]

and \[ \log 74584 = 4.8726457 \ldots \ldots \ldots (2). \]

The quantity \( \log 74583.3 \) will clearly lie between \( \log 74583 \) and \( \log 74584. \)

Let then \[ \log 74583.3 = \log 74583 + x \]

\[ = 4.8726398 + x \ldots \ldots \ldots (3). \]
From (1) and (2), we see that for an increase 1 in the number the increase in the logarithm is 0.000059.

The Theory of Proportional Parts then states that for an increase of 3 in the number the increase in the logarithm is 

$$3 \times 0.000059, \text{ i.e., } 0.0000177.$$  

Hence \(\log 74583.3 = 4.8726398 + 0.0000177\)

\[= 4.87264157.\]

**149.** As another example, we shall find the value of \(\log 0.0382757\) and shall exhibit the working in a more concise form.

From the tables we obtain

\[
\begin{align*}
\log 0.038275 &= 2.5829152 \\
\log 0.038276 &= 2.5829265.
\end{align*}
\]

Hence the difference for

\[0.00001 = 0.000113.\]

Therefore the difference for

\[0.000007 = 0.7 \times 0.000113\]

\[= 0.0000791.\]

\[\therefore \log 0.0382757 = 2.5829152 + 0.0000791\]

\[= 2.58292311.\]

Since we only require logarithms to seven places of decimals, we omit the last digit and the answer is \(2.5829231\).

**150.** The converse question is often met with, viz., to find the number whose logarithm is given. If the logarithm be one of those tabulated the required number is easily found. The method to be followed when this is not the case is shewn in the following examples.
Find the number whose logarithm is 2.6283924.

On reference to the tables we find that the logarithm 6283924 is not tabulated, but that the nearest logarithms are 6283889 and 6283991, between which our logarithm lies.

We have then
\[ \log 425.00 = 2.6283889 \] ..........(1),
and
\[ \log 425.01 = 2.6283991 \] ..........(2).
Let
\[ \log (425.00 + x) = 2.6283924 \] ...... ...... ...... (3).

From (1) and (2), we see that corresponding to a difference .01 in the number there is a difference .0000102 in the logarithm.

From (1) and (3), we see that corresponding to a difference \(x\) in the number there is a difference .0000035 in the logarithm.

Hence we have \[ x : .01 :: .0000035 : .0000102. \]
\[ \therefore x = \frac{3.5}{102} \times .01 = \frac{35}{102} = .00343 \text{ nearly.} \]

Hence the required number = 425.00 + .00343 = 425.00343.

151. Where logarithms are taken out of the tables the labour of subtracting successive logarithms may be avoided. On reference to page 153 there is found at the extreme right a column headed Diff. The number 82 at the head of the figures in this column gives the difference corresponding to a difference unity in the numbers on that page.

This number 82 means .0000082.

The rows below the 82 give the differences corresponding to .1, .2,... Thus the fifth of these rows means that the difference for .5 is .0000041.

As an example, let us find the logarithm of 52746.74.
From page 153, we have
\[ \log 52746 = 4.7221895 \]
diff. for \[ \cdot 7 = .0000057 \]
diff. for \[ \cdot 04 \]
\[ \left(= \frac{1}{10} \times \text{diff. for} \ 4\right) = .0000003 \]
\[ \therefore \log 52746.74 = 4.7221955. \]

L. T.
We shall solve two more examples, taking all the logarithms from the tables, and only putting down the necessary steps.

**Ex. 1.** Find the seventh root of 0.031574.

If $x$ be the required quantity, we have

$$\log x = \frac{1}{7} \log (0.031574) = \frac{1}{7} (2.5387496)$$

$$= \frac{1}{7} (7 + 5.5387496).$$

:. $\log x = 1.7912499.$

But

$$\log 61837 = 1.7912484$$

diff. = 0.000013.

But diff. for 0.0001 = 0.000071,

:. required increase = 0.0000211,

:. $x = 61837211.$

**Ex. 2.** If $a = 34562.73$ and $b = 28347.912$, find the value of the square root of $a^2 - b^2$.

If $x$ be the required quantity, we have

$$2 \log x = \log (a^2 - b^2) = \log (a - b) + \log (a + b)$$

$$= \log 6214.818 + \log 62910.642.$$

Now

$$\log 6214.8 = 3.7934272$$

1 7

8 5 6

$\log 62910.6 = 4.7987197$ 14

6 41 4 28

2

Hence, by addition, $2 \log x = 8.592152554$.

:. $\log x = 4.2960763.$

But

$$\log 19773 = 4.2960726$$

:. diff. = 37.

But

$\text{diff. for } 1 = 220,$

:. proportional increase = $\frac{37}{220} \times 1 = 0.168$,

:. $x = 19773.168.$

152. The proof of the Principle of Proportional Parts will not be given at this stage. It is not strictly true without certain limitations.
The numbers to which the principle is applied must contain not less than five significant figures, and then we may rely on the result as correct to seven places of decimals.

For example, we must not apply the principle to obtain the value of log 2.5 from the values of log 2 and log 3.

For, if we did, since these logarithms are .30103 and .4771213, the logarithm of 2.5 would be .389075.

But from the tables the value of log 2.5 is found to be .3979400.

Hence the result which we should obtain would be manifestly quite incorrect.

Tables of trigonometrical ratios.

153. In Chambers' Tables will be found tables giving the values of the trigonometrical ratios of angles between 0° and 45°, the angles increasing by differences of 1'.

It is unnecessary to separately tabulate the ratios for angles between 45° and 90°, since the ratios of angles between 45° and 90° can be reduced to those of angles between 0° and 45°. (Art. 75.)

For example,

\[ \sin 76° 11' = \sin (90° - 13° 49') = \cos 13° 49', \]

and is therefore known).

Such a table is called a table of natural sines, cosines, etc. to distinguish it from the table of logarithmic sines, cosines, etc.

If we want to find the sine of an angle which contains an integral number of degrees and minutes, we can obtain
it from the tables. If, however, the angle contain seconds, we must use the principle of proportional parts.

**Ex. 1.** Given \( \sin 29^\circ 14'=0.4883674 \),

and \( \sin 29^\circ 15'=0.4886212 \),

**find the value of \( \sin 29^\circ 14'32'' \).**

By subtraction we have

\[
\text{difference in the sine for } 1' = 0.0002538.
\]

\[
\therefore \text{difference in the sine for } 32'' = \frac{32}{60} \times 0.0002538 = 0.00013536,
\]

\[
\therefore \sin 29^\circ 14'32'' = 0.4883674 + 0.00013536 = 0.48850276.
\]

Since we want our answer only to seven places of decimals, we omit the last 6, and, since 76 is nearer to 80 than 70, we write

\[
\sin 29^\circ 14'32'' = 0.4885028.
\]

**N.B.** When we omit a figure in the eighth place of decimals we add 1 to the figure in the seventh place, if the omitted figure be 5 or a number greater than 5.

**Ex. 2.** Given \( \cos 16^\circ 27'=0.9590672 \),

and \( \cos 16^\circ 28'=0.9589848 \),

**find \( \cos 16^\circ 27'47'' \).**

We note that, as was shown in Art. 55, the cosine decreases as the angle increases.

Hence for an increase of 1', i.e. 60'', in the angle, there is a decrease of \( 0.0000824 \) in the cosine.

Hence for an increase of 47'' in the angle, there is a decrease of \( \frac{47}{60} \times 0.0000824 \) in the cosine.

\[
\therefore \cos 16^\circ 27'47'' = 0.9590672 - \frac{47}{60} \times 0.0000824
\]

\[
= 0.9590672 - 0.0000645
\]

\[
= 0.9590672
\]

\[
- 0.0000645
\]

\[
= 0.9590027.
\]
In practice this may be abbreviated thus:

\[
\begin{align*}
cos 16^\circ 28' &= \cdot9589843 \\
cos 16^\circ 27' &= \cdot9590672 \\
\text{diff. for } 1' &= -\cdot000824. \\
\therefore \text{diff. for } 47'' &= -\frac{1}{47} \times -\cdot000824 \\
&= -\cdot0000615. \\
\therefore \text{Ans.} &= \cdot9590672 \\
&\quad -\cdot0000615 \\
&= \cdot9590027.
\end{align*}
\]

154. The inverse question, to find the angle, when one of its trigonometrical ratios is given, will now be easy.

**Ex.** Find the angle whose cotangent is 1.410325, having given \(\cot 35^\circ 19'=1.4114799\), and \(\cot 35^\circ 20'=1.4106098\).

Let the required angle be \(35^\circ 19' + x''\),

so that \(\cot (35^\circ 19' + x'') = 1.410325\).

From these three equations we have

For an increase of 60'' in the angle, a decrease of \(\cdot0008701\) in the cotangent,

\[
\begin{align*}
\therefore x : 60 &\propto 5141 : 8701, \text{ so that } x=37.7.
\end{align*}
\]

Hence the required angle = \(35^\circ 19' 37.7''\).

155. In working all questions involving the application of the Principle of Proportional Parts, the student must be very careful to note whether the trigonometrical ratios increase or decrease as the angle increases. As a help to his memory, he may observe that in the first quadrant the three trigonometrical ratios whose names begin with co-, i.e. the cosine, the cotangent, and the cosecant, all decrease as the angle increases.
Tables of logarithmic sines, cosines, etc.

156. In many kinds of trigonometric calculation, as in the solution of triangles, we often require the logarithms of trigonometrical ratios. To avoid the inconvenience of first finding the sine of any angle from the tables and then obtaining the logarithm of this sine by a second application of the tables, it has been found desirable to have separate tables giving the logarithms of the various trigonometrical functions of angles. As before, it is only necessary to construct the tables for angles between $0^\circ$ and $45^\circ$.

Since the sine of an angle is always less than unity, the logarithm of its sine is always negative (Art. 142).

Again, since the tangent of an angle between $0^\circ$ and $45^\circ$ is less than unity its logarithm is negative, whilst the logarithm of the tangent of an angle between $45^\circ$ and $90^\circ$ is the logarithm of a number greater than unity and is therefore positive.

157. To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trigonometric functions, the logarithms as tabulated are not the true logarithms, but the true logarithms increased by 10.

For example, sine $30^\circ = \frac{1}{2}$.

Hence \[\log \sin 30^\circ = \log \frac{1}{2} = -\log 2\]

\[= -0.30103 = 1.69897.\]

The logarithm tabulated is therefore

\[10 + \log \sin 30^\circ, i.e. \ 9.69897.\]

Again, \[\tan 60^\circ = \sqrt{3}.\]
Hence \( \log \tan 60^\circ = \frac{1}{2} \log 3 = \frac{1}{2} (\cdot4771213) \)
\[ = \cdot2385606. \]

The logarithm tabulated is therefore
\[ 10 + \cdot2385606, \textit{i.e.} 10\cdot2385606. \]

The symbol \( L \) is used to denote these "tabular logarithms," \textit{i.e.} the logarithms as found in the English books of tables.

Thus \( L \sin 15^\circ 25' = 10 + \log \sin 15^\circ 25' \),
and \( L \sec 48^\circ 23' = 10 + \log \sec 48^\circ 23' \).

\textbf{158.} If we want to find the tabular logarithm of any function of an angle, which contains an integral number of degrees and minutes, we can obtain it directly from the tables. If, however, the angle contain seconds we must use the principle of proportional parts. The method of procedure is similar to that of Art. 153. We give an example and also one of the inverse question.

\textbf{Ex. 1.} Given \( L \cosec 32^\circ 21' = 10\cdot2715733 \),
and \( L \cosec 32^\circ 22' = 10\cdot2713740 \),
find \( L \cosec 32^\circ 21' 51'' \).

For an increase of 60" in the angle, there is a \textit{decrease} of \( \cdot0001993 \) in the logarithm.
Hence for an increase of 51" in the angle, the corresponding \textit{decrease} is \( \frac{51}{60} \times \cdot0001993 \), \textit{i.e.} \( \cdot0001694 \).

Hence \( L \cosec 32^\circ 21' 51'' = 10\cdot2715733 \)
\[ - \cdot0001694 \]
\[ = 10\cdot2714039. \]

\textbf{Ex. 2.} \textit{Find the angle such that the tabular logarithm of its tangent is 9.4417250.}

Let \( x \) be the required angle.
From the tables, we have

\[
\begin{align*}
L \tan x &= 9.4417250 \\
L \tan 15^\circ 27' &= 9.4415145
\end{align*}
\]

\[
\text{diff.} = 2105.
\]

\[
\begin{align*}
L \tan 15^\circ 28' &= 9.4420062 \\
L \tan 15^\circ 27' &= 9.4415145
\end{align*}
\]

\[
\text{diff. for } 1' = 4917.
\]

Corresponding increase \( \frac{2105}{60} \times 60'' = 25.7'' \),

\[\therefore x = 15^\circ 27' 25.7''.\]

**Ex. 3.** Given \( L \sin 14^\circ 6' = 9.3867040 \), find \( L \cosec 14^\circ 6' \).

Here \( \log \sin 14^\circ 6' = L \sin 11^\circ 6' - 10 \)

\[= -1 + 9.3867040.\]

Now \( \log \cosec 14^\circ 6' = \log \frac{1}{\sin 14^\circ 6'} \)

\[= -\log \sin 14^\circ 6'
= 1 - 9.3867040 = 6132960.\]

Hence \( L \cosec 14^\circ 6' = 10.6132960 \).

More generally, we have \( \sin \theta \times \cosec \theta = 1 \).

\[\therefore \log \sin \theta + \log \cosec \theta = 0.\]

\[\therefore L \sin \theta + L \cosec \theta = 20.\]

The error to be avoided is this; the student sometimes assumes that, because

\[\log \cosec 14^\circ 6' = -\log \sin 14^\circ 6',\]

he may therefore assume that

\[L \cosec 14^\circ 6' = - L \sin 14^\circ 6'.\]

This is obviously untrue.

**EXAMPLES. XXIV.**

1. **Given** \( \log 35705 = 4.5527290 \)

and \( \log 35706 = 4.5527412 \),

find the values of \( \log 35705.7 \) and \( \log 35.70585 \).

2. **Given** \( \log 5.8742 = 0.7689487 \)

and \( \log 587.43 = 2.7689561 \),

find the values of \( \log 58742.57 \) and \( \log 0.00587422 \).
3. Given \( \log 47847 = 4.6798547 \)
and \( \log 47848 = 4.6798638 \),
find the numbers whose logarithms are respectively
\( 2.6798593 \) and \( 3.6798617 \).

4. Given \( \log 258.36 = 2.4122253 \)
and \( \log 2.5837 = .4122421 \),
find the numbers whose logarithms are
\( .4122378 \) and \( 2.4122287 \).

5. From the table on page 153 find the logarithms of
   (1) \( 52538.97 \), (2) \( 527.236 \), (3) \( 000529673 \),
and the numbers whose logarithms are
   (4) \( 3.7221098 \), (5) \( 2.7210075 \), and (6) \( .7210386 \).

6. Given \( \sin 43^\circ 23' = .6868761 \)
and \( \sin 43^\circ 24' = .6870875 \),
find the value of \( \sin 43^\circ 23' 47'' \).

7. Find also the angle whose sine is \( .6870349 \).

8. Given \( \cos 32^\circ 16' = .8455726 \)
and \( \cos 32^\circ 17' = .8454172 \),
find the values of \( \cos 32^\circ 16' 24'' \) and of \( \cos 32^\circ 16' 47'' \).

9. Find also the angles whose cosines are
   \( \cdot8454832 \) and \( \cdot8455176 \).

10. Given \( \tan 76^\circ 21' = 4.1177784 \)
and \( \tan 76^\circ 22' = 4.1230079 \),
find the values of \( \tan 76^\circ 21' 29'' \) and \( \tan 76^\circ 21' 47'' \).

11. Given \( \cosec 13^\circ 8' = 4.4010616 \)
and \( \cosec 13^\circ 9' = 4.3955817 \),
find the values of \( \cosec 13^\circ 8' 19'' \) and \( \cosec 13^\circ 8' 37'' \).

12. Find also the angle whose cosecant is \( 4.396789 \).

13. Given \( L \cos 34^\circ 44' = 9.9147729 \)
and \( L \cos 34^\circ 45' = 9.9146852 \),
find the value of \( L \cos 34^\circ 44' 27'' \).
14. Find also the angle $\theta$, where

$$L \cos \theta = 9.9147328.$$ 

15. Given

$$L \cot 71^\circ 27' = 9.5257779$$
and

$$L \cot 71^\circ 28' = 9.5253580,$$ find the value of

$$L \cot 71^\circ 27' 47'',$$
and solve the equation

$$L \cot \theta = 9.5254782.$$ 

16. Given

$$L \sec 18^\circ 27' = 10.0229163$$
and

$$L \sec 18^\circ 28' = 10.0229590,$$ find the value of

$$L \sec 18^\circ 27' 35''.$$ 

17. Find also the angle whose $L \sec$ is $10.0229285$. 

18. Find in degrees, minutes, and seconds the angle whose sine is $0.6$, given that

$$\log 6 = 7781513, L \sin 36^\circ 52' = 9.7781186,$$
and

$$L \sin 36^\circ 53' = 9.7782870.$$ 

159. On the next page is printed a specimen page taken from Chambers' tables. It gives the tabular logarithms of the ratios of angles between $32^\circ$ and $33^\circ$ and also between $57^\circ$ and $58^\circ$.

The first column gives the $L$ sine for each minute between $32^\circ$ and $33^\circ$.

In the second column under the word Diff. is found the number 2021. This means that $0.002021$ is the difference between $L \sin 32^\circ 0'$ and $L \sin 32^\circ 1'$; this may be verified by subtracting $9.7242097$ from $9.7244118$. It will also be noted that the figures 2021 are printed halfway between the numbers $9.7242097$ and $9.7244118$, thus clearly shewing between what numbers it is the difference.

This same column of Differences also applies to the column on its right-hand side which is headed Cosec.

Similarly the fifth column, which is also headed Diff., may be used with the two columns on the right and left of it.
|------|-------|--------|-------|-------|---------|--------|-------|--------|------|
160. There is one point to be noticed in using the columns headed Diff. It has been pointed out that 2021 (at the top of the second column) means \( \cdot0002021 \). Now the 790 (at the top of the eighth column) means not \( \cdot000790 \), but \( \cdot0000790 \). The rule is this; the right-hand figure of the Diff. must be placed in the seventh place of decimals and the requisite number of cyphers prefixed. Thus

\[
\begin{align*}
\text{Diff.} &= 74 \quad \Rightarrow \quad 9 \quad \Rightarrow \quad \cdot0000009, \quad \text{(9 means that the difference is \( \cdot0000009 \))} \\
\text{Diff.} &= 735 \quad \Rightarrow \quad \cdot000074, \\
\text{Diff.} &= 2021 \quad \Rightarrow \quad \cdot000735, \\
\text{ whilst Diff.} &= 12348 \quad \Rightarrow \quad \cdot0012348.
\end{align*}
\]

161. Page 171 also gives the tabular logs. of ratios between 57° and 58°. Suppose we wanted \( L \tan 57° 20' \). We now start with the line at the bottom of the page and run our eye up the column which has Tang. at its foot. We go up this column until we arrive at the number which is on the same level as the number 20 in the extreme right-hand column. This number we find to be \(101930286\), which is therefore the value of

\[
L \tan 57° 20'.
\]

EXAMPLES. XXV.

1. Find \( \theta \), given that \( \cos \theta = \cdot9725382 \), \( \cos 13° 27' = \cdot9725733 \), diff. for \( 1' = 677 \).

2. Find the angle whose sine is \( \frac{3}{8} \), given \( \sin 22° 1' = \cdot3748763 \), diff. for \( 1' = 2636 \).
3. Given \( \cosec 65^\circ 24' = 1.0998243 \),
   diff. for \( 1' = 1464 \),
find the value of \( \cosec 65^\circ 24' 37'' \),
and the angle whose cosec is \( 1.0997938 \).

4. Given \( L \tan 22^\circ 37' = 9.6197205 \),
   diff. for \( 1' = 3557 \),
find the value of \( L \tan 22^\circ 37' 22'' \),
and the angle whose \( L \tan \) is \( 9.6195283 \).

5. Find the angle whose \( L \cos \) is \( 9.993 \), given
   \( L \cos 10^\circ 15' = 9.9930131 \), diff. for \( 1' = 229 \).

6. Find the angle whose \( L \sec \) is \( 10.15 \), given
   \( L \sec 44^\circ 55' = 10.1498843 \), diff. for \( 1' = 1200 \).

7. From the table on page 171 find the values of
   (1) \( L \sin 32^\circ 18' 23'' \),
   (2) \( L \cos 32^\circ 16' 49'' \),
   (3) \( L \cot 32^\circ 29' 43'' \),
   (4) \( L \sec 32^\circ 52' 27'' \),
   (5) \( L \tan 57^\circ 45' 28'' \),
   (6) \( L \cosec 57^\circ 48' 21'' \),
and
   (7) \( L \cos 57^\circ 58' 29'' \).

8. With the help of the same page solve the equations
   (1) \( L \tan \theta = 10.1959261 \),
   (2) \( L \cosec \theta = 10.0738125 \),
   (3) \( L \cos \theta = 9.9259283 \), and (4) \( L \sin \theta = 9.9241352 \).

9. Take out of the tables \( L \tan 16^\circ 6' 23'' \) and calculate the value of
the square root of the tangent.

10. Change into a form more convenient for logarithmic computation
    (i.e. express in the form of products of quantities) the quantities
    (1) \( 1 + \tan x \tan y \),
    (2) \( 1 - \tan x \tan y \),
    (3) \( \cot x + \tan y \),
    (4) \( \cot x - \tan y \),
    (5) \( \frac{1 - \cos 2x}{1 + \cos 2x} \), and (6) \( \frac{\tan x + \tan y}{\cot x + \cot y} \).
CHAPTER XII

RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRICAL RATIOs OF THE ANGLES OF ANY TRIANGLE.

162. In any triangle $ABC$, the side $BC$, opposite to the angle $A$, is denoted by $a$; the sides $CA$ and $AB$, opposite to the angles $B$ and $C$ respectively, are denoted by $b$ and $c$.

163. **Theorem.** In any triangle $ABC$,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. the sines of the angles are proportional to the opposite sides.

![Diagram]

Draw $AD$ perpendicular to the opposite side meeting it, produced if necessary, in the point $D$. 
In the triangle $ABD$, we have
\[
\frac{AD}{AB} = \sin B, \text{ so that } AD = c \sin B.
\]

In the triangle $ACD$, we have
\[
\frac{AD}{AC} = \sin C, \text{ so that } AD = b \sin C.
\]

[If the angle $C$ be obtuse, as in the second figure, we have
\[
\frac{AD}{b} = \sin ACD = \sin (180^\circ - C) = \sin C \quad \text{(Art. 72)},
\]
so that $AD = b \sin C$.]

Equating these two values of $AD$, we have
\[
c \sin B = b \sin C,
\]
i.e.
\[
\frac{\sin B}{b} = \frac{\sin C}{c}.
\]

In a similar manner, by drawing a perpendicular from $B$ upon $CA$, we have
\[
\frac{\sin C}{c} = \frac{\sin A}{a}.
\]

If one of the angles, $C$, be a right angle, as in the third figure, we have $\sin C = 1$,
\[
\sin A = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.
\]

Hence
\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}.
\]

We therefore have, in all cases,
\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
\]
CHAPTER XII.

RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF ANY TRIANGLE.

162. In any triangle $ABC$, the side $BC$, opposite to the angle $A$, is denoted by $a$; the sides $CA$ and $AB$, opposite to the angles $B$ and $C$ respectively, are denoted by $b$ and $c$.

163. Theorem. In any triangle $ABC$,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. the sines of the angles are proportional to the opposite sides.

Draw $AD$ perpendicular to the opposite side meeting it, produced if necessary, in the point $D$. 
In the triangle $ABD$, we have
\[
\frac{AD}{AB} = \sin B, \text{ so that } AD = c \sin B.
\]

In the triangle $ACD$, we have
\[
\frac{AD}{AC} = \sin C, \text{ so that } AD = b \sin C.
\]

[If the angle $C$ be obtuse, as in the second figure, we have
\[
\frac{AD}{b} = \sin ACD = \sin (180^\circ - C) = \sin C \quad \text{(Art. 72),}
\]
so that \( AD = b \sin C \).

Equating these two values of $AD$, we have
\[c \sin B = b \sin C,
\]
i.e.
\[
\frac{\sin B}{b} = \frac{\sin C}{c}.
\]

In a similar manner, by drawing a perpendicular from $B$ upon $CA$, we have
\[
\frac{\sin C}{c} = \frac{\sin A}{a}.
\]

If one of the angles, $C$, be a right angle, as in the third figure, we have $\sin C = 1$,
\[
\sin A = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.
\]

Hence
\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}.
\]

We therefore have, in all cases,
\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
\]
164. In any triangle, to find the cosine of an angle in terms of the sides.

Let $ABC$ be the triangle and let the perpendicular from $A$ on $BC$ meet it, produced if necessary, in the point $D$.

First, let the angle $C$ be acute, as in the first figure.

By Euc. II. 13, we have

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD \ldots \ldots \ldots \ldots (i).$$

But $\frac{CD}{CA} = \cos C$, so that $CD = b \cos C$.

Hence (i) becomes

$$c^2 = a^2 + b^2 - 2a \cdot b \cos C,$$

i.e.

$$2ab \cos C = a^2 + b^2 - c^2,$$

i.e.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Secondly, let the angle $C$ be obtuse, as in the second figure.

By Euc. II. 12, we have

$$\dot{A}B^2 = BC^2 + CA^2 + 2BC \cdot CD \ldots \ldots (ii).$$

But $\frac{CD}{CA} = \cos ACD = \cos (180^\circ - C) = - \cos C$,

so that $CD = -b \cos C$. (Art. 72)
Hence (ii) becomes
\[ c^2 = a^2 + b^2 + 2a(-b \cos C) = a^2 + b^2 - 2ab \cos C, \]
so that, as in the first case, we have
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab}. \]

In a similar manner it may be shown that
\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc}, \]
and
\[ \cos B = \frac{c^2 + a^2 - b^2}{2ca}. \]

If one of the angles, \( C \), be a right angle, the above formula would give \( c^2 = a^2 + b^2 \), so that \( \cos C = 0 \). This is correct, since \( C \) is a right angle.

The above formula is therefore true for all values of \( C \).

**Ex.** If \( a = 15, \ b = 36, \) and \( c = 39, \)
then \[ \cos A = \frac{36^2 + 39^2 - 15^2}{2 \times 36 \times 39} = \frac{3^2(12^2 + 13^2 - 5^2)}{2 \times 3^2 \times 12 \times 13} = \frac{288}{24 \times 13} = \frac{12}{13}. \]

165. To find the sines of half the angles in terms of the sides.

In any triangle we have, by Art. 164,
\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} \]
By Art. 109, we have
\[ \cos A = 1 - 2 \sin^2 \frac{A}{2} \]
Hence \[ 2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} \]
\[ = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = \frac{a^2 - (b - c)^2}{2bc} \]
\[ = \frac{[a + (b - c)][a - (b - c)]}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc} \]...(1).

L.T. 12
Let $2s$ stand for $a + b + c$, so that $s$ is equal to half the sum of the sides of the triangle, i.e. $s$ is equal to the semi-perimeter of the triangle.

We then have

\[a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c),\]
and \[a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b).\]

The relation (1) therefore becomes

\[2 \sin^2 \frac{A}{2} = \frac{2(s - c) \times 2(s - b)}{2bc} = \frac{(s - b)(s - c)}{bc}. \]

\[\therefore \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad \ldots \ldots \ldots (2). \]

Similarly,

\[\sin \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{ca}}, \text{ and } \sin \frac{C}{2} = \sqrt{\frac{(s - a)(s - b)}{ab}}. \]

166. To find the cosines of half the angles in terms of the sides.

By Art. 109, we have

\[\cos A = 2 \cos^2 \frac{A}{2} - 1. \]

Hence \[2 \cos^2 \frac{A}{2} = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} \]

\[= \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc} \]

\[= \frac{[(b + c) + a][b + c - a]}{2bc} = \frac{(b + c)(b + c - a)}{2bc} \quad \ldots \ldots (1). \]

Now \[b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a), \]

so that (1) becomes

\[ 2 \cos^2 \frac{A}{2} = \frac{s^2 \times 2(s - a)}{2bc} = \frac{s(s - a)}{bc}. \]

\[ \therefore \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}} \quad \text{..................(2)}. \]

Similarly,

\[ \cos \frac{B}{2} = \sqrt{\frac{s(s - b)}{ca}}, \quad \text{and} \quad \cos \frac{C}{2} = \sqrt{\frac{s(s - c)}{ab}}. \]

167. To find the tangents of half the angles in terms of the sides.

Since

\[ \tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}, \]

we have, by (2) of Arts. 165 and 166,

\[ \tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \div \sqrt{\frac{s(s - a)}{bc}} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \]

Similarly,

\[ \tan \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{s(s - b)}}, \quad \text{and} \quad \tan \frac{C}{2} = \sqrt{\frac{(s - a)(s - b)}{s(s - c)}}. \]

Since, in a triangle, \( A \) is always \(< 180^\circ\), \( \frac{A}{2} \) is always \(< 90^\circ\).

The sine, cosine, and tangent of \( \frac{A}{2} \) are therefore always positive (Art. 52).

The positive sign must therefore always be prefixed to the radical sign in the formulae of this and the last two articles.
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168. Ex. \[ a=13, \ b=14, \text{ and } c=15, \]
then \[ s=\frac{13+14+15}{2}=21, \ s-a=8, \ s-b=7, \]
and \[ s-c=6. \]
Hence \[ \sin \frac{A}{2} = \sqrt{\frac{7 \times 6}{14 \times 15}} = \frac{1}{5} \sqrt{5}, \]
\[ \sin \frac{B}{2} = \sqrt{\frac{6 \times 8}{15 \times 13}} = \frac{4}{65} \sqrt{65}, \]
\[ \cos \frac{C}{2} = \sqrt{\frac{21 \times 6}{13 \times 14}} = \frac{3}{13} \sqrt{13}, \]
and \[ \tan \frac{B}{2} = \sqrt{\frac{6 \times 8}{21 \times 7}} = \frac{4}{7}. \]

169. To express the sine of any angle of a triangle in terms of the sides.
We have, by Art. 109,
\[ \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}. \]

But, by the previous articles,
\[ \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ and } \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \]
Hence
\[ \sin A = 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}. \]
\[ \therefore \sin A = \frac{2}{bc} \sqrt{s (s-a) (s-b) (s-c)}. \]

EXAMPLES. XXVI.

In a triangle
1. Given \[ a=25, \ b=52, \text{ and } c=63, \]
find \[ \tan \frac{A}{2}, \tan \frac{B}{2}, \text{ and } \tan \frac{C}{2}. \]
2. Given \( a = 125, \ b = 123, \) and \( c = 62, \)
find the sines of half the angles and the sines of the angles.

3. Given \( a = 18, \ b = 24, \) and \( c = 30, \)
find \( \sin A, \ \sin B, \) and \( \sin C. \)
Verify by a graph.

4. Given \( a = 35, \ b = 84, \) and \( c = 91, \)
find \( \tan A, \ \tan B, \) and \( \tan C. \)

5. Given \( a = 13, \ b = 14, \) and \( c = 15, \)
find the sines of the angles. Verify by a graph.

6. Given \( a = 287, \ b = 816, \) and \( c = 865, \)
find the values of \( \tan \frac{A}{2} \) and \( \tan A. \)

7. Given \( a = \sqrt{3}, \ b = \sqrt{2}, \) and \( c = \frac{\sqrt{6} + \sqrt{2}}{2}, \)
find the angles.

170. In any triangle, to prove that,
\[ a = b \cos C + c \cos B. \]

Take the figures of Art. 164.
In the first case, we have
\[ \frac{BD}{BA} = \cos B, \] so that \( BD = c \cos B, \)
and \[ \frac{CD}{CA} = \cos C, \] so that \( CD = b \cos C. \)
Hence \( a = BC = BD + DC = c \cos B + b \cos C. \)
In the second case, we have
\[ \frac{BD}{BA} = \cos B, \] so that \( BD = c \cos B, \)
and \[ \frac{CD}{CA} = \cos \angle ACD = \cos (180^\circ - C) \]
\[ = - \cos C \text{ (Art. 72)}, \]
so that \( CD = - b \cos C. \)
Hence, in this case,

\[ a = BC = BD - CD = c \cos B - (-b \cos C), \]

so that in each case

\[ a = b \cos C + c \cos B. \]

Similarly, \[ b = c \cos A + a \cos C, \]

and \[ c = a \cos B + b \cos A. \]

**171. In any triangle, to prove that**

\[ \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}. \]

In any triangle, we have

\[ \frac{b}{c} = \frac{\sin B}{\sin C}. \]

\[ \therefore \frac{b - c}{b + c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}}{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}} \]

\[ = \frac{\tan \frac{B - C}{2}}{\tan \frac{B + C}{2}} = \frac{\tan \frac{B - C}{2}}{\tan \left( 90^\circ - \frac{A}{2} \right)} \]

\[ = \frac{\tan \frac{B - C}{2}}{\cot \frac{A}{2}} \quad \text{(Art. 69).} \]

Hence \[ \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}. \]
172. Ex. From the formulae of Art. 164 deduce those of Art. 170 and vice versa.

The first and third formulae of Art. 164 give

\[ b \cos C + c \cos B = \frac{a^2 + b^2 - c^2}{2a} + \frac{c^2 + a^2 - b^2}{2a} = \frac{2a^2}{2a} = a, \]

so that \( a = b \cos C + c \cos B \).

Similarly, the other formulae of Art. 170 may be obtained.

Again, the three formulae of Art. 170 give

\[ a = b \cos C + c \cos B, \]
\[ b = c \cos A + a \cos C, \]
\[ c = a \cos B + b \cos A. \]

Multiplying these in succession by \( a, b, \) and \(-c\) we have, by addition,

\[ a^2 + b^2 - c^2 = a(b \cos C + c \cos B) + b(c \cos A + a \cos C) - c(a \cos B + b \cos A) = 2ab \cos C. \]

\[ \therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}. \]

Similarly, the other formulae of Art. 162 may be found.

173. The student will often meet with identities, which he is required to prove, which involve both the sides and the angles of a triangle.

It is, in general, desirable in the identity to substitute for the sides in terms of the angles, or to substitute for the ratios of the angles in terms of the sides.

Ex. 1. Prove that \( a \cos \frac{B - C}{2} = (b + c) \sin \frac{A}{2} \).

By Art. 163, we have

\[ \frac{b + c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\cos \frac{A}{2} \cos \frac{B - C}{2}}{\sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\cos \frac{B - C}{2}}{\sin \frac{A}{2}}. \]

\[ \therefore (b + c) \sin \frac{A}{2} = a \cos \frac{B - C}{2}. \]
Ex. 2. In any triangle prove that

\[(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.\]

By Art. 163 we have

\[\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}.\]

Hence the given expression

\[= (b^2 - c^2) \frac{\cos A}{ak} + (c^2 - a^2) \frac{\cos B}{bk} + (a^2 - b^2) \frac{\cos C}{ck}\]

\[= k \left[ (b^2 - c^2) \frac{b^2 + c^2 - a^2}{2abc} + (c^2 - a^2) \frac{c^2 + a^2 - b^2}{2abc} + (a^2 - b^2) \frac{a^2 + b^2 - c^2}{2abc} \right]\]

\[= \frac{1}{2abc} \left[ b^4 - c^4 - a^2 (b^2 - c^2) + c^4 - a^4 - b^2 (c^2 - a^2) + a^4 - b^4 - c^2 (a^2 - b^2) \right]\]

\[= 0.\]

Ex. 3. In any triangle prove that

\[(a + b + c) \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) = 2c \cot \frac{C}{2}.\]

The left-hand member

\[= 2s \left[ \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \right], \text{ by Art. 167,}\]

\[= 2s \sqrt{\frac{s-c}{s}} \left[ \sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}} \right] = 2s \sqrt{\frac{s-c}{s}} \frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\]

\[= 2\sqrt{s(s-c)} \cdot \frac{c}{\sqrt{(s-a)(s-b)}}, \text{ since } 2s = a + b + c,\]

\[= 2c \cot \frac{C}{2}.\]

This identity may also be proved by substituting for the sides.

We have, by Art. 163,

\[\frac{a + b + c}{c} = \frac{\sin A + \sin B + \sin C}{\sin C}\]

\[= \frac{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}}, \text{ as in Art. 127,} = \frac{2 \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{C}{2}}.\]
Also
\[ \frac{2 \cot \frac{C}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}} = \frac{2 \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{C}{2} \left( \sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2} \right)} \]

\[ = \frac{2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{C}{2} \sin \frac{A+B}{2}} = \frac{2 \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{C}{2}}. \]  

(Art. 69.)

We have therefore

\[ \frac{a+b+c}{e} = \frac{2 \cot \frac{C}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}}, \]

so that

\[ (a+b+c) \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) = 2c \cot \frac{C}{2}. \]

**Ex. 4.** *If the sides of a triangle be in Arithmetical Progression, prove that so also are the cotangents of half the angles.*

We have given that \( a+c=2b \) ............................................(1),

and we have to prove that \( \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2} \) ............................................(2).

Now (2) is true if

\[ \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 2 \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}, \]

or, by multiplying both sides by

\[ \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \]

if \( (s-a) + (s-c) = 2(s-b), \)

i.e. if \( 2s - (a+c) = 2s - 2b, \)

i.e. if \( a+c=2b, \) which is relation (1).

Hence if relation (1) be true, so also is relation (2).
EXAMPLES. XXVII.

In any triangle $ABC$, prove that

1. $\sin \frac{B - C}{2} = \frac{b - c}{a} \cos \frac{A}{2}$.

2. $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$.

3. $a \left( b \cos C - c \cos B \right) = b^2 - c^2$.

4. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

5. $a \left( \cos B + \cos C \right) = 2(b + c) \sin^2 \frac{A}{2}$.

6. $a \left( \cos C - \cos B \right) = 2(b - c) \cos^2 \frac{A}{2}$.

7. $\frac{\sin (B - C)}{\sin (B + C)} = \frac{b^2 - c^2}{a^2}$.

8. $\frac{a + b}{a - b} = \tan \frac{A + B}{2} \cot \frac{A - B}{2}$.

9. $a \sin \left( \frac{A}{2} + B \right) = (b + c) \sin \frac{A}{2}$.

10. $\frac{a^2 \sin (B - C)}{\sin B + \sin C} + \frac{b^2 \sin (C - A)}{\sin C + \sin A} + \frac{c^2 \sin (A - B)}{\sin A + \sin B} = 0$.

11. $(b + c - a) \left( \cot \frac{B}{2} + \cot \frac{C}{2} \right) = 2a \cot \frac{A}{2}$.

12. $a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C)$.

13. $(a^2 - b^2 + c^2) \tan B = (a^2 + b^2 - c^2) \tan C$.

14. $c^2 = (a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$.

15. $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$.

16. $\frac{a \sin (B - C)}{b^2 - c^2} = \frac{b \sin (C - A)}{c^2 - a^2} = \frac{c \sin (A - B)}{a^2 - b^2}$.

17. $a \sin \frac{A}{2} \sin \frac{B - C}{2} + b \sin \frac{B}{2} \sin \frac{C - A}{2} + c \sin \frac{C}{2} \sin \frac{A - B}{2} = 0$. 
18. \( a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0. \)

19. \( \frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0. \)

20. \( \frac{(a+b+c)^2}{a^2 + b^2 + c^2} = \frac{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}{\cot A + \cot B + \cot C}. \)

21. \( a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3 abc. \)

22. In a triangle whose sides are 3, 4, and \( \sqrt{38} \) feet respectively, prove that the largest angle is greater than 120°.

23. The sides of a right-angled triangle are 21 and 28 feet; find the length of the perpendicular drawn to the hypotenuse from the right angle.

24. If in any triangle the angles be to one another as 1 : 2 : 3, prove that the corresponding sides are as 1 : \( \sqrt{3} : 2. \)

25. In any triangle, if \( \tan \frac{A}{2} = \frac{5}{6} \) and \( \tan \frac{B}{2} = \frac{20}{37}, \) find \( \tan \frac{C}{2}, \) and prove that in this triangle \( a+c=2b. \)

26. In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Shew that it divides the angle into parts whose cotangents are 2 and 3.

27. The perpendicular \( AD \) to the base of a triangle \( ABC \) divides it into segments such that \( BD, CD, \) and \( AD \) are in the ratio of 2, 3, and 6; prove that the vertical angle of the triangle is 45°.

28. A ring, ten inches in diameter, is suspended from a point one foot above its centre by 6 equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings.

29. If \( a^2, b^2, \) and \( c^2 \) be in A.P., prove that \( \cot A, \cot B, \) and \( \cot C \) are in A.P. also.

30. If \( a, b, \) and \( c \) be in A.P., prove that \( \cos A \cot \frac{A}{2}, \cos B \cot \frac{B}{2}, \) and \( \cos C \cot \frac{C}{2} \) are in A.P.
31. If $a$, $b$, and $c$ are in H.P., prove that $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, and $\sin^2 \frac{C}{2}$ are also in H.P.

32. The sides of a triangle are in A.P. and the greatest and least angles are $\theta$ and $\phi$; prove that
\[ 4 (1 - \cos \theta)(1 - \cos \phi) = \cos \theta + \cos \phi. \]

33. The sides of a triangle are in A.P. and the greatest angle exceeds the least by $90^\circ$; prove that the sides are proportional to $\sqrt{7} + 1$, $\sqrt{7}$, and $\sqrt{7} - 1$.

34. If $C = 60^\circ$, then prove that
\[ \frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}. \]

35. In any triangle $ABC$ if $D$ be any point of the base $BC$, such that $BD:DC::m:n$, and if $\angle BAD = a$, $\angle DAC = \beta$, $\angle CD A = \theta$, and $AD = x$, prove that
\[ (m+n) \cot \theta = m \cot a - n \cot \beta = n \cot B - m \cot C, \]
and
\[ (m+n)^2 \cdot x^2 = (m+n) (mb^2 + nc^2) - mna^2. \]

36. If in a triangle the bisector of the side $c$ be perpendicular to the side $b$, prove that
\[ 2 \tan A + \tan C = 0. \]

37. In any triangle prove that, if $\theta$ be any angle, then
\[ b \cos \theta = c \cos (A - \theta) + a \cos (C + \theta). \]

38. If $p$ and $q$ be the perpendiculars from the angular points $A$ and $B$ on any line passing through the vertex $C$ of the triangle $ABC$, then prove that
\[ a^2p^2 + b^2q^2 - 2abpq \cos C = a^2b^2 \sin^2 C. \]

39. In the triangle $ABC$, lines $OA$, $OB$, and $OC$ are drawn so that the angles $OAB$, $OBC$, and $OCA$ are each equal to $\omega$; prove that
\[ \cot \omega = \cot A + \cot B + \cot C, \]
and
\[ \csc^2 \omega = \csc^2 A + \csc^2 B + \csc^2 C. \]
CHAPTER XIII.

SOLUTION OF TRIANGLES.

174. In any triangle the three sides and the three angles are often called the elements of the triangle. When any three elements of the triangle are given, provided they be not the three angles, the triangle is in general completely known, i.e. its other angles and sides can be calculated. When the three angles are given, only the ratios of the lengths of the sides can be found, so that the triangle is given in shape only and not in size. When three elements of a triangle are given the process of calculating its other three elements is called the Solution of the Triangle.

We shall first discuss the solution of right-angled triangles, i.e. triangles which have one angle given equal to a right angle.

The next four articles refer to such triangles, and $C$ denotes the right angle.

175. Case I. Given the hypothenuse and one side, to solve the triangle.
Let \( b \) be the given side and \( c \) the given hypotenuse. The angle \( B \) is given by the relation
\[
\sin B = \frac{b}{c}.
\]
\[
\therefore \ L \sin B = 10 + \log b - \log c.
\]
Since \( b \) and \( c \) are known, we thus have \( L \sin B \) and therefore \( B \).
The angle \( A \) \( (= 90^\circ - B) \) is then known. The side \( a \) is obtained from either of the relations
\[
\cos B = \frac{a}{c}, \ \tan B = \frac{b}{a}, \ \text{or} \ a = \sqrt{(c - b)(c + b)}.
\]

176. Case II. Given the two sides \( a \) and \( b \), to solve the triangle.
Here \( B \) is given by
\[
\tan B = \frac{b}{a},
\]
so that
\[
L \tan B = 10 + \log b - \log a.
\]
Hence \( L \tan B \), and therefore \( B \), is known.
The angle \( A \) \( (= 90^\circ - B) \) is then known.
The hypotenuse \( c \) is given by the relation \( c = \sqrt{a^2 + b^2} \).
This relation is not however very suitable for logarithmic calculation, and \( c \) is best given by
\[
\sin B = \frac{b}{c}, \ \text{i.e.} \ c = \frac{b}{\sin B}.
\]
\[
\therefore \ \log c = \log b - \log \sin B
\]
\[
= 10 + \log b - L \sin B.
\]
Hence \( c \) is obtained.
177. Case III. Given an angle $B$ and one of the sides $a$, to solve the triangle.

Here $A (= 90^\circ - B)$ is known.

The side $b$ is found from the relation

$$\frac{b}{a} = \tan B,$$

and $c$ from the relation

$$\frac{a}{c} = \cos B.$$

178. Case IV. Given an angle $B$ and the hypothenuse $c$, to solve the triangle.

Here $A$ is known, and $a$ and $b$ are obtained from the relations

$$\frac{a}{c} = \cos B,$$

and

$$\frac{b}{c} = \sin B.$$

EXAMPLES. XXVIII.

1. In a right-angled triangle $ABC$, where $C$ is the right angle, if $a=50$ and $B=75^\circ$, find the sides. ($\tan 75^\circ = 2+\sqrt{3}$.)

2. Solve the triangle of which two sides are equal to 10 and 20 feet and of which the included angle is $90^\circ$; given that $\log 20=1.30103$, and $L \tan 26^\circ 33'=9.6986847$, diff. for $1'=3160$.

3. The length of the perpendicular from one angle of a triangle upon the base is 3 inches and the lengths of the sides containing this angle are 4 and 5 inches. Find the angles, having given

$$\log 2=0.30103, \log 3=0.4771213,$$

$$L \sin 36^\circ 52'=9.7781186, \text{ diff. for } 1'=1684,$$

and

$$L \sin 48^\circ 35'=9.8750142, \text{ diff. for } 1'=1115.$$

4. Find the acute angles of a right-angled triangle whose hypothenuse is four times as long as the perpendicular drawn to it from the opposite angle.
179. We now proceed to the case of the triangle which is not given to be right angled.

The different cases to be considered are;

Case I. The three sides given;

Case II. Two sides and the included angle given;

Case III. Two sides and the angle opposite one of them given;

Case IV. One side and two angles given;

Case V. The three angles given.

180. Case I. The three sides $a$, $b$, and $c$ given.

Since the sides are known, the semi-perimeter $s$ is known and hence also the quantities $s-a$, $s-b$, and $s-c$.

The half-angles $\frac{A}{2}$, $\frac{B}{2}$, and $\frac{C}{2}$ are then found from the formulae

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

and

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Only two of the angles need be found, the third being known since the sum of the three angles is always 180°.

The angles may also be found by using the formulae for the sine or cosine of the semi-angles.

(Arts. 165 and 166.)

The above formulae are all suited for logarithmic computation.
THE THREE SIDES GIVEN.

The angle \( A \) may also be obtained from the formula

\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc}.
\]

(Art. 164.)

This formula is not, in general, suitable for logarithmic calculation. It may be conveniently used however when the sides \( a \), \( b \), and \( c \) are small numbers.

**Ex.** The sides of a triangle are 32, 40, and 66 feet; find the angle opposite the greatest side, having given that

\[
\log 207 = 2.3159703, \log 1073 = 3.0305997,
\]

\[
L \cot 66^\circ 18' = 9.6124311, \text{ tabulated difference for } 1' = 3431.
\]

Here \( a = 32, \ b = 40, \) and \( c = 66, \)

so that \( s = \frac{32 + 40 + 66}{2} = 69, \ s - a = 37, \ s - b = 29, \) and \( s - c = 3. \)

Hence \( \cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{69 \times 3}{37 \times 29}} = \sqrt{\frac{207}{1073}}. \)

\[
L \cot \frac{C}{2} = 10 + \frac{1}{2} [\log 207 - \log 1073]
\]

\[
= 10 + 1.15798515 - 1.51529985
\]

\[
= 9.6426853.
\]

\( L \cot \frac{C}{2} \) is therefore greater than \( L \cot 66^\circ 18', \)

so that \( \frac{C}{2} \) is less than \( 66^\circ 18'. \)

Let then \( \frac{C}{2} = 66^\circ 18' - \alpha'. \)

The difference in the logarithm corresponding to difference of \( \alpha' \) in the angle therefore

\[
= 9.6426853 - 9.6424341 = 0.0002512.
\]

Also the difference for \( 60'' = 0.0003431. \)

**L. T.** 13
Hence \[ \frac{x}{60} = \frac{0.002512}{0.0005431}, \]
so that \[ x = \frac{2512}{3431} \times 60 = \text{nearly 14}. \]

\[ \therefore \frac{C}{2} = 60^\circ 18' - 44'' = 60^\circ 17'16'', \text{ and hence } C = 132^\circ 34'32''. \]

**EXAMPLES. XXIX.**

*The student should verify the results of some of the following examples (e.g. Nos. 1, 7, 8, 10, 11, 12) by an accurate graph.*

1. If the sides of a triangle be 56, 65, and 33 feet, find the greatest angle.

2. The sides of a triangle are 7, 4\sqrt{3}, and \sqrt{13} yards respectively. Find the number of degrees in its smallest angle.

3. The sides of a triangle are \(x^2 + x + 1, 2x + 1, \text{ and } x^2 - 1\); prove that the greatest angle is 120°.

4. The sides of a triangle are \(a, b, \text{ and } \sqrt{a^2 + ab + b^2}\) feet; find the greatest angle.

5. If \(a = 2, \ b = \sqrt{6}, \text{ and } c = \sqrt{3} - 1, \) solve the triangle.

6. If \(a = 2, \ b = \sqrt{6}, \text{ and } c = \sqrt{3} + 1, \) solve the triangle.

7. If \(a = 9, \ b = 10, \text{ and } c = 11, \) find \(B, \) given

\[ \log 2 = 0.30103, \quad L \tan 29^\circ 29' = 9.7523472, \]

and

\[ L \tan 29^\circ 30' = 9.7526420. \]

8. The sides of a triangle are 130, 123, and 77 feet. Find the greatest angle, having given

\[ \log 2 = 0.30103, \quad L \tan 38^\circ 39' = 9.9029376, \]

and

\[ L \tan 38^\circ 40' = 9.9031966. \]

9. Find the greatest angle of a triangle whose sides are 212, 188, and 270 feet, having given

\[ \log 2 = 0.30103, \quad \log 3 = 0.4771213, \quad \log 7 = 0.8450980, \]

\[ L \tan 38^\circ 20' = 9.8980104, \text{ and } L \tan 38^\circ 19' = 9.8977507. \]

10. The sides of a triangle are 2, 3, and 4; find the greatest angle, having given

\[ \log 2 = 0.30103, \quad \log 3 = 0.4771213, \]

\[ L \tan 52^\circ 14' = 10.1108395, \]

and

\[ L \tan 52^\circ 15' = 10.1111004. \]
Making use of the tables, find all the angles when

11. \( a = 25, \ b = 26, \) and \( c = 27. \)

12. \( a = 17, \ b = 20, \) and \( c = 27. \)

13. \( a = 2000, \ b = 1050, \) and \( c = 1150. \)

181. Case II. Given two sides \( b \) and \( c \) and the included angle \( A. \)

Taking \( b \) to be the greater of the two given sides, we have

\[
\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2} \quad (\text{Art. 171}) \quad \ldots (1),
\]

and

\[
\frac{B + C}{2} = 90^\circ - \frac{A}{2} \quad \ldots \ldots (2).
\]

These two relations give us

\[
\frac{B - C}{2} \quad \text{and} \quad \frac{B + C}{2},
\]

and therefore, by addition and subtraction, \( B \) and \( C. \)

The third side \( a \) is then known from the relation

\[
\frac{a}{\sin A} = \frac{b}{\sin B},
\]

which gives

\[
a = b \frac{\sin A}{\sin B},
\]

and thus determines \( a. \)

The side \( a \) may also be found from the formula

\[
a^2 = b^2 + c^2 - 2bc \cos A.
\]

This is not adapted to logarithmic calculation but is sometimes useful, especially when the sides \( a \) and \( b \) are small numbers.
182. **Ex. 1.** If \( b = \sqrt{3} \), \( c = 1 \), and \( A = 30^\circ \), solve the triangle.

We have

\[
\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cot 15^\circ.
\]

Now

\[
\tan 15^\circ = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \quad (\text{Art. 101}),
\]

so that

\[
\cot 15^\circ = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.
\]

Hence

\[
\tan \frac{B - C}{2} = 1.
\]

\[
\therefore \quad \frac{B - C}{2} = 45^\circ \quad \text{.......................... (1)}.
\]

Also

\[
\frac{B + C}{2} = 90^\circ - \frac{A}{2} = 90^\circ - 15^\circ = 75^\circ \quad \text{................. (2)}.
\]

By addition, \( B = 120^\circ \).

By subtraction, \( C = 30^\circ \).

Since \( A = C \), we have \( a = c = 1 \).

**Otherwise.** We have

\[
a^2 = b^2 + c^2 - 2bc \cos A = 3 + 1 - 2 \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 1,
\]

so that

\[
a = 1 = c.
\]

\[
\therefore \quad C = A = 30^\circ,
\]

and

\[
B = 180^\circ - A - C = 120^\circ.
\]

**Ex. 2.** If \( b = 215 \), \( c = 105 \), and \( A = 74^\circ 27' \), find the remaining angles and also the third side \( a \), having given

\[
\log 2 = 0.3010300, \quad \log 11 = 1.0413927,
\]

\[
\log 105 = 2.0211893, \quad \log 212.476 = 2.3273103,
\]

\[
L \cot 37^\circ 13' = 10.1194723, \quad \text{diff. for } 1' = 2622,
\]

\[
L \tan 24^\circ 20' = 9.6553477, \quad \text{diff. for } 1' = 3364,
\]

\[
L \sin 74^\circ 27' = 0.98838052,
\]

and

\[
L \cosec 28^\circ 25' = 10.3225025, \quad \text{diff. for } 1' = 2334.
\]
SOLUTION OF TRIANGLES.

Here \( \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2} = \frac{1}{2} \cot 37^\circ 13' 30'' \).

Now \( L \cot 37^\circ 13' = 10.1194723 \)

\[
\begin{align*}
\text{diff. for } 30'' &= -1311 \\
\therefore L \cot 37^\circ 13' 30'' &= 10.1193412 \\
\log 11 &= 1.0413927 \\
\hline
\log 32 &= 1.50515 \\
\therefore L \tan \frac{1}{2} (B - C) &= 9.6555839 \\
\text{But } L \tan 24^\circ 20' &= 9.6553477 \\
\text{diff. } &= 2362 \\
\therefore \frac{B - C}{2} &= 24^\circ 20' 42''.
\end{align*}
\]

\[
\text{But } \frac{B + C}{2} = 90^\circ - \frac{A}{2} = 52^\circ 46' 30''.
\]

\( \therefore \) by addition, \( B = 77^\circ 7' 12'' \),
and, by subtraction, \( C = 28^\circ 25' 48''. \)

Again \( \frac{a}{\sin A} = \frac{c}{\sin C} = c \cosec C \),

\( \therefore a = 105 \sin 74^\circ 27' \cosec 28^\circ 25' 48''. \)

\[
\begin{align*}
\text{But } L \cosec 28^\circ 25' &= 10.3225025 \\
\text{diff. for } 48'' &= -1867 \\
\hline
L \cosec 28^\circ 25' 48'' &= 10.3223158 \\
L \sin 74^\circ 27' &= 9.9838052 \\
\log 105 &= 2.0211893 \\
\hline
22.3273103 \\
\hline
\therefore \log a &= 2.3273103, \\
\therefore a &= 212.476.
\end{align*}
\]
183. There are ways of finding the third side $a$ of the triangle in the previous case without first finding the angles $B$ and $C$.

Two methods are as follows:

1. Since
\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ = b^2 + c^2 - 2bc \left( 2 \cos^2 \frac{A}{2} - 1 \right) \]
\[ = (b + c)^2 - 4bc \cos^2 \frac{A}{2}. \]
\[ \therefore a^2 = (b + c)^2 \left[ 1 - \frac{4bc}{(b + c)^2} \cos^2 \frac{A}{2} \right]. \]

Hence, if
\[ \sin^2 \theta = \frac{4bc}{(b + c)^2} \cos^2 \frac{A}{2}, \]
we have
\[ a^2 = (b + c)^2 \left[ 1 - \sin^2 \theta \right] = (b + c)^2 \cos^2 \theta, \]
so that
\[ a = (b + c) \cos \theta. \]

If then $\sin \theta$ be calculated from the relation
\[ \sin \theta = \frac{2 \sqrt{bc}}{b + c} \cos \frac{A}{2}, \]
we have
\[ a = (b + c) \cos \theta. \]

2. We have
\[ a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 - 2bc \left( 1 - \sin^2 \frac{A}{2} \right) \]
\[ = (b - c)^2 + 4bc \sin^2 \frac{A}{2} \]
\[ = (b - c)^2 \left[ 1 + \frac{4bc}{(b - c)^2} \sin^2 \frac{A}{2} \right]. \]

Let
\[ \frac{4bc}{(b - c)^2} \sin^2 \frac{A}{2} = \tan^2 \phi, \]
so that
\[ \tan \phi = \frac{2 \sqrt{bc}}{b - c} \sin \frac{A}{2}, \]
and hence $\phi$ is known.

Then
\[ a^2 = (b - c)^2 \left[ 1 + \tan^2 \phi \right] = \frac{(b - c)^2}{\cos^2 \phi}, \]
so that
\[ a = (b - c) \sec \phi, \]
and is therefore easily found.

An angle, such as $\theta$ or $\phi$ above, introduced for the purpose of facilitating calculation is called a subsidiary angle (Art. 129).
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EXAMPLES. XXX.

[The student should verify the results of some of the following examples (e.g. Nos. 4, 5, 6, 11) by an accurate graph.]

1. If \( b = 90^\circ \), \( c = 70 \), and \( A = 72^\circ 48' 30'' \), find \( B \) and \( C \), given

\[
\log 2 = 30103, \quad L \cot 36^\circ 24' 15'' = 10.1323111,
\]

\[
L \tan 9^\circ 37' = 9.2290071,
\]

and

\[
L \tan 9^\circ 38' = 9.2297735.
\]

2. If \( a = 21 \), \( b = 11 \), and \( C = 34^\circ 42' 30'' \), find \( A \) and \( B \), given

\[
\log 2 = 30103,
\]

and

\[
L \tan 72^\circ 38' 45'' = 10.50515.
\]

3. If the angles of a triangle be in A. P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the length of the third side and the angles, given

\[
\log 2 = 30103, \quad \log 3 = 4771213,
\]

and

\[
L \tan 19^\circ 6' = 9.5394287, \quad \text{diff. for } 1' = 4084.
\]

4. If \( a = 13 \), \( b = 7 \), and \( C = 60^\circ \), find \( A \) and \( B \), given that

\[
\log 3 = 4771213,
\]

and

\[
L \tan 27^\circ 27' = 9.7155508, \quad \text{diff. for } 1' = 3087.
\]

5. If \( a = 2b \), and \( C = 120^\circ \), find the values of \( A, B, \) and the ratio of \( c \) to \( a \), given that

\[
\log 3 = 4771213,
\]

and

\[
L \tan 10^\circ 53' = 9.2839070, \quad \text{diff. for } 1' = 6808.
\]

6. If \( b = 14 \), \( c = 11 \), and \( A = 60^\circ \), find \( B \) and \( C \), given that

\[
\log 2 = 30103, \quad \log 3 = 4771213,
\]

\[
L \tan 11^\circ 44' = 9.3174299,
\]

and

\[
L \tan 11^\circ 45' = 9.3180640.
\]

7. The two sides of a triangle are 510 and 420 yards long respectively and include an angle of \( 52^\circ 6' \). Find the remaining angles, given that

\[
\log 2 = 30103, \quad L \tan 26^\circ 3' = 9.6891430,
\]

\[
L \tan 14^\circ 20' = 9.4074189, \quad \text{and } L \tan 14^\circ 21' = 9.4079453.
\]
8. If \( b = 2.5 \text{ ft.}, c = 2 \text{ ft.}, \text{ and } A = 22° 20', \) find the other angles, and show that the third side is nearly one foot, given
\[
\log 2 = 30103, \quad \log 3 = .47712,
\]
\[
L \cot 11° 10' = 10.70465, \quad L \sin 22° 20' = 9.57977,
\]
\[
L \tan 29° 22' 20'' = 9.75038, \quad L \tan 29° 22' 30'' = 9.75013,
\]
and
\[
L \sin 49° 27' 34'' = 9.88079.
\]

9. If \( a = 2, b = 1 + \sqrt{3}, \text{ and } C = 60°, \) solve the triangle.

10. Two sides of a triangle are \( \sqrt{3} + 1 \) and \( \sqrt{3} - 1, \) and the included angle is 60°; find the other side and angles.

11. If \( b = 1, c = \sqrt{3} - 1, \text{ and } A = 60°, \) find the length of the side \( a. \)

12. If \( b = 91, c = 125, \text{ and } \tan \frac{A}{2} = \frac{17}{6}, \) prove that \( a = 204. \)

13. If \( a = 5, b = 4, \text{ and } \cos (A - B) = \frac{31}{32}, \) prove that the third side \( c \) will be 6.

14. One angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and 40 \( \sqrt{3} \) yards. Find the length of the third side and the number of degrees in the other angles.

15. The sides of a triangle are 9 and 3, and the difference of the angles opposite to them is 90°. Find the base and the angles, having given
\[
\log 2 = 30103, \quad \log 3 = .4771213,
\]
\[
\log 75894 = 4.8802074, \quad \log 75895 = 4.8802132,
\]
and
\[
L \tan 26° 33' = 9.6986847,
\]
and
\[
L \tan 26° 34' = 9.6990006.
\]

16. If
\[
\tan \phi = \frac{a - b}{a + b} \cot \frac{C}{2},
\]
prove that
\[
c = (a + b) \frac{\sin \frac{C}{2}}{\cos \phi}.
\]

If \( a = 3, b = 1, \text{ and } C = 53° 7'48'', \) find \( c \) without getting \( A \) and \( B, \)
given
\[
\log 2 = 30103, \quad \log 25298 = 4.4030862,
\]
\[
\log 25299 = 4.4031034, \quad L \cos 26° 33' 54'' = 9.9515452,
\]
and
\[
L \tan 26° 33' 54'' = 9.6889700.
17. Two sides of a triangle are 237 and 158 feet and the contained angle is $66^\circ 40'$; find the base and the other angles, having given

$$\log 2 = .30103, \log 79 = 1.89763,$$

$$\log 22687 = 4.35578, L \cot 33^\circ 20' = 10.18197,$$

$$L \sin 33^\circ 20' = 9.73998, L \tan 16^\circ 54' = 9.48262,$$

$$L \tan 16^\circ 55' = 9.48308, L \sec 16^\circ 54' = 10.01917,$$

and

$$L \sec 16^\circ 55' = 10.01921.$$

[Use either the formula $\cos \frac{B - C}{2} = \frac{b + c}{a} \sin \frac{A}{2}$ or the formula of the preceding question.]

In the following four examples, the required logarithms must be taken from the tables.

18. If $a = 242.5$, $b = 164.3$, and $C = 54^\circ 36'$, solve the triangle.

19. If $b = 130$, $c = 63$, and $A = 42^\circ 15' 30''$, solve the triangle.

20. Two sides of a triangle being 2265.4 and 1779 feet, and the included angle $58^\circ 17'$, find the remaining angles.

21. Two sides of a triangle being 237.09 and 130.96 feet, and the included angle $57^\circ 59'$, find the remaining angles.

184. Case III. Given two sides $b$ and $c$ and the angle $B$ opposite to one of them.

The angle $C$ is given by the relation

$$\frac{\sin C}{c} = \frac{\sin B}{b},$$

i.e. $$\sin C = \frac{c}{b} \sin B \ldots \ldots (1).$$

Taking logarithms, we determine $C$, and then $A (= 180^\circ - B - C)$ is found.
The remaining side $a$ is then found from the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

i.e.

$$a = b \frac{\sin A}{\sin B} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2).$$

185. The equation (1) of the previous article gives in some cases no value, in some cases one, and sometimes two values, for $C$.

First, let $B$ be an acute angle.

(a) If $b < c \sin B$, the right-hand member of (1) is greater than unity, and hence there is no corresponding value for $C$.

(β) If $b = c \sin B$, the right-hand member of (1) is equal to unity and the corresponding value of $C$ is $90^\circ$.

(γ) If $b > c \sin B$, there are two values of $C$ having $\frac{c \sin B}{b}$ as its sine, one value lying between $0^\circ$ and $90^\circ$ and the other between $90^\circ$ and $180^\circ$.

Both of these values are not however always admissible. For if $b > c$, then $B > C$. The obtuse-angled value of $C$ is now not admissible; for, in this case, $C$ cannot be obtuse unless $B$ be obtuse also, and it is manifestly impossible to have two obtuse angles in a triangle.

If $b < c$ and $B$ be an acute angle, both values of $C$ are admissible. Hence there are two values found for $A$, and hence the relation (2) gives two values for $a$. In this case there are therefore two triangles satisfying the given conditions.

Secondly, let $B$ be an obtuse angle.

If $b$ be $< or = c$, then $B$ would be less than, or equal
to, C, so that C would be an obtuse angle. The triangle would then be impossible.

If b be > c, the acute value of C, as determined from (1), would be admissible, but not the obtuse value. We should therefore only have one admissible solution.

Since, for some values of b, c and B, there is a doubt or ambiguity in the determination of the triangle, this case is called the **Ambiguous Case** of the solution of triangles.

186. The Ambiguous Case may also be discussed in a geometrical manner.

Suppose we were given the elements b, c, and B, and that we proceeded to construct, or attempted to construct, the triangle.

We first measure an angle $ABD$ equal to the given angle $B$.

We then measure along $BA$ a distance $BA$ equal to the given distance c, and thus determine the angular point A.

We have now to find a third point C, which must lie on $BD$ and must also be such that its distance from A shall be equal to b.

To obtain it, we describe with centre A a circle whose radius is b.

The point or points, if any, in which this circle meets $BD$ will determine the position of C.

Draw $AD$ perpendicular to $BD$, so that

$$AD = AB \sin B = c \sin B.$$  

One of the following events will happen.

The circle may not reach $BD$ (Fig. 1) or it may
touch $BD$ (Fig. 2), or it may meet $BD$ in two points $C_1$ and $C_2$ (Figs. 3 and 4).

In the case of Fig. 1, it is clear that there is no triangle satisfying the given condition.

Here $b < AD$, i.e. $< c \sin B$.

In the case of Fig. 2, there is one triangle $ABD$ which is right-angled at $D$. Here

$$b = AD = c \sin B.$$  

In the case of Fig. 3, there are two triangles $ABC_1$ and $ABC_2$. Here $b$ lies in magnitude between $AD$ and $c$, i.e. $b$ is $> c \sin B$ and $< c$.

In the case of Fig. 4, there is only one triangle $ABC_1$ satisfying the given conditions [the triangle $ABC_2$ is inadmissible; for its angle at $B$ is not equal to $B$ but is equal to $180^\circ - B$]. Here $b$ is greater than both $c \sin B$ and $c$. 

---

**Fig. 1**

**Fig. 2**

**Fig. 3**

**Fig. 4**
In the case when $B$ is obtuse, the proper figures should be drawn. It will then be seen that when $b < c$ there is no triangle (for in the corresponding triangles $ABC_1$ and $ABC_2$ the angle at $B$ will be $180^\circ - B$ and not $B$). If $b > c$, it will be seen that there is one triangle, and only one, satisfying the given conditions.

**To sum up:**

Given the elements $b$, $c$, and $B$ of a triangle,

(a) If $b$ be $< c \sin B$, there is no triangle.

(b) If $b = c \sin B$, there is one triangle right-angled.

(γ) If $b$ be $> c \sin B$ and $< c$ and $B$ be acute, there are two triangles satisfying the given conditions.

(δ) If $b$ be $> c$, there is only one triangle.

Clearly if $b = c$, the points $B$ and $C_2$ in Fig. 3 coincide and there is only one triangle.

(ε) If $B$ be obtuse, there is no triangle except when $b > c$.

187. The ambiguous case may also be considered algebraically as follows.

From the figure of Art. 184, we have

$$b^2 = c^2 + a^2 - 2ca \cos B.$$  

$$\therefore \ a^2 - 2ac \cos B + c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B$$  

$$= b^2 - c^2 \sin^2 B.$$  

$$\therefore \ a - c \cos B = \pm \sqrt{b^2 - c^2 \sin^2 B},$$  

i.e.  

$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B} \ldots \ldots \ldots (1).$$

Now (1) is an equation to determine the value of $a$ when $b$, $c$, and $B$ are given.
(α) If \( b < c \sin B \), the quantity \( \sqrt{b^2 - c^2 \sin^2 B} \) is imaginary, and (1) gives no real value for \( a \).

(β) If \( b = c \sin B \), there is only one value, \( c \cos B \), for \( a \); there is thus only one triangle which is right-angled.

(γ) If \( b > c \sin B \), there are two values for \( a \). But, since \( a \) must be positive, the value obtained by taking the lower sign affixed to the radical is inadmissible unless

\[
c \cos B - \sqrt{b^2 - c^2 \sin^2 B} \text{ is positive},
\]

i.e. unless \( \sqrt{b^2 - c^2 \sin^2 B} < c \cos B \),

i.e. unless \( b^2 - c^2 \sin^2 B < c^2 \cos^2 B \),

i.e. unless \( b^2 < c^2 \).

There are therefore two triangles only when \( b \) is \( > c \sin B \) and at the same time \( < c \).

(δ) If \( B \) be an obtuse angle, then \( c \cos B \) is negative, and one value of \( a \) is always negative and the corresponding triangle impossible.

The other value will be positive only when

\[
c \cos B + \sqrt{b^2 - c^2 \sin^2 B} \text{ is positive},
\]

i.e. only when \( \sqrt{b^2 - c^2 \sin^2 B} > - c \cos B \),

i.e. only when \( b^2 > c^2 \sin^2 B + c^2 \cos^2 B \),

i.e. only when \( b > c \).

Hence, \( B \) being obtuse, there is no triangle if \( b < c \), and only one triangle when \( b > c \).

188. \textbf{Ex.} Given \( b = 16 \), \( c = 25 \), and \( B = 33^\circ 15' \), prove that the triangle is ambiguous and find the other angles, having given

\[
\begin{align*}
\log 2 &= 0.30103, & L \sin 33^\circ 15' &= 0.7390129, \\
L \sin 38^\circ 56' &= 0.9327616, \\
\text{and} & & L \sin 58^\circ 57' &= 0.9328376.
\end{align*}
\]
SOLUTION OF TRIANGLES.

We have
\[
\sin C = \frac{c}{b} \sin B = \frac{25}{10} \sin 100^\circ = \frac{10^2}{36} \sin 33^\circ 15'.
\]
Hence
\[
L \sin C = 2 + L \sin 33^\circ 15' - 6 \log 2 = 9:9328329.
\]
Hence
\[
\begin{align*}
L \sin C &= 9:9328329 \\
L \sin 58^\circ 57' &= 9:9328376 \\
L \sin 58^\circ 56' &= 9:9327616 \\
\text{Diff.} &= 713 \\
\text{Diff. for } 1' &= 760.
\end{align*}
\]
\[
\therefore \text{angular diff.} = \frac{713}{6} \times 60'' = 56'' \text{ nearly.}
\]
\[
\therefore C = 58^\circ 56' 56'' \text{ or } 180^\circ - 58^\circ 56' 56''.
\]
Hence (Fig. 3, Art. 186) we have
\[
C_1 = 58^\circ 56' 56'', \text{ and } C_2 = 121^\circ 3' 4''.
\]
\[
\therefore \angle BAC_1 = 180^\circ - 33^\circ 15' - 58^\circ 56' 56'' = 87^\circ 48' 4'',
\]
and
\[
\angle BAC_2 = 180^\circ - 33^\circ 15' - 121^\circ 3' 4'' = 25^\circ 41' 56''.
\]

EXAMPLES. XXXI.

[The student should verify the results of some of the following examples (e.g. Nos. 3, 5, 6, 8, 9, 10, 12, 13) by an accurate graph.]

1. If \(a = 5\), \(b = 7\), and \(\sin A = \frac{3}{4}\), is there any ambiguity?

2. If \(a = 2\), \(c = \sqrt{3} + 1\), and \(A = 45^\circ\), solve the triangle.

3. If \(a = 100\), \(c = 100\sqrt{2}\), and \(A = 30^\circ\), solve the triangle.

4. If \(2b = 3a\), and \(\tan^2 A = \frac{3}{5}\), prove that there are two values to the third side, one of which is double the other.

5. If \(A = 30^\circ\), \(b = 8\), and \(a = 6\), find \(c\).

6. Given \(B = 30^\circ\), \(c = 150\), and \(b = 50\sqrt{3}\), prove that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the greater value of the third side.

Would the solution have been ambiguous had
\[
B = 30^\circ, c = 150, \text{ and } b = 75?\]
7. In the ambiguous case given \(a\), \(b\), and \(A\), prove that the difference between the two values of \(c\) is \(2\sqrt{a^2 - b^2 \sin^2 A}\).

8. If \(a = 5\), \(b = 4\), and \(A = 45^\circ\), find the other angles, having given

\[
\log 2 = 0.30103, \quad L \sin 33^\circ 29' = 9.7520507,
\]

and

\[
L \sin 33^\circ 30' = 9.7530993.
\]

9. If \(a = 9\), \(b = 12\), and \(A = 30^\circ\), find \(c\), having given

\[
\log 2 = 0.30103, \quad \log 3 = 0.47712,
\]

\[
\log 171 = 2.23301, \quad \log 368 = 2.56635,
\]

\[
L \sin 11^\circ 48' 39'' = 9.31108, \quad L \sin 41^\circ 48' 39'' = 9.82391,
\]

and

\[
L \sin 108^\circ 11' 21'' = 9.97774.
\]

10. Point out whether or not the solutions of the following triangles are ambiguous.

Find the smaller value of the third side in the ambiguous case and the other angles in both cases.

(1) \(A = 30^\circ\), \(c = 250\) feet, and \(a = 125\) feet;

(2) \(A = 30^\circ\), \(c = 250\) feet, and \(a = 200\) feet.

Given

\[
\log 2 = 0.30103, \quad \log 6.03893 = 0.7809601,
\]

\[
L \sin 38^\circ 41' = 9.7958800;
\]

and

\[
L \sin 8^\circ 41' = 9.1789001.
\]

11. Given \(a = 250\), \(b = 240\), and \(A = 72^\circ 4' 48''\), find the angles \(B\) and \(C\), and state whether they can have more than one value, given

\[
\log 2.5 = 0.3979400, \quad \log 2.4 = 0.3802112,
\]

\[
L \sin 72^\circ 4' = 9.9783702, \quad L \sin 72^\circ 5' = 9.9734111,
\]

and

\[
L \sin 65^\circ 59' = 9.9606739.
\]

12. Two straight roads intersect at an angle of 30\(^\circ\); from the point of junction two pedestrians \(A\) and \(B\) start at the same time, \(A\) walking along one road at the rate of 5 miles per hour and \(B\) walking uniformly along the other road. At the end of 3 hours they are 9 miles apart. Shew that there are two rates at which \(B\) may walk to fulfil this condition and find them.
For the following three examples, a book of tables will be required.

13. Two sides of a triangle are 1015 feet and 732 feet, and the angle opposite the latter side is 40°; find the angle opposite the former and prove that more than one value is admissible.

14. Two sides of a triangle being 5374·5 and 1586·6 feet, and the angle opposite the latter being 15°11′, calculate the other angles of the triangle or triangles.

15. Given \(A = 10°\), \(a = 2308·7\), and \(b = 7903·2\), find the smaller value of \(c\).

189. Case IV. Given one side and two angles, viz. \(a\), \(B\), and \(C\).

Since the three angles of a triangle are together equal to two right angles, the third angle is given also.

The sides \(b\) and \(c\) are now obtained from the relations

\[
\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A},
\]

giving

\[b = a \frac{\sin B}{\sin A}, \text{ and } c = a \frac{\sin C}{\sin A}.\]

190. Case V. The three angles \(A\), \(B\), and \(C\) given.

Here the ratios only of the sides can be determined by the formulae

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

Their absolute magnitudes cannot be found.
EXAMPLES. XXXII.

1. If \( \cos A = \frac{17}{22} \) and \( \cos C = \frac{1}{14} \), find the ratio of \( a : b : c \).

2. The angles of a triangle are as 1 : 2 : 7; prove that the ratio of the greatest side to the least side is \( \sqrt{5} + 1 : \sqrt{5} - 1 \).

3. If \( A = 45^\circ \), \( B = 75^\circ \), and \( C = 60^\circ \), prove that \( a + c \sqrt{2} = 2b \).

4. Two angles of a triangle are \( 41^\circ 13' 22'' \) and \( 71^\circ 19' 5'' \) and the side opposite the first angle is 55; find the side opposite the latter angle, given

\[
\log 55 = 1.7403627, \quad \log \frac{79063}{4.8979775}, \\
L \sin 41^\circ 13' 22'' = 0.8188779, \\
L \sin 71^\circ 19' 5'' = 0.9764927.
\]

5. From each of two ships, one mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be \( 52^\circ 25' 15'' \) and \( 75^\circ 9' 30'' \) respectively. Given

\[
L \sin 75^\circ 9' 30'' = 0.9852635, \\
L \sin 52^\circ 25' 15'' = 0.8990055, \quad \log 1.2197 = 0.0862530, \\
\log 1.2198 = 0.0862886,
\]

find the distance of the beacon from each of the ships.

6. The base angles of a triangle are \( 22\frac{1}{2}^\circ \) and \( 112\frac{1}{2}^\circ \); prove that the base is equal to twice the height.

For the following five questions a book of tables is required.

7. The base of a triangle being seven feet and the base angles \( 123^\circ 23' \) and \( 38^\circ 36' \), find the length of its shorter side.

8. If the angles of a triangle be as 5 : 10 : 21, and the side opposite the smaller angle be 3 feet, find the other sides.

9. The angles of a triangle being \( 150^\circ \), \( 18^\circ 20' \), and \( 11^\circ 40' \), and the longest side being 1000 feet, find the length of the shortest side.

10. To get the distance of a point \( A \) from a point \( B \), a line \( BC \) and the angles \( ABC \) and \( BCA \) are measured, and are found to be 287 yards and \( 55^\circ 32' 10'' \) and \( 51^\circ 8' 20'' \) respectively. Find the distance \( AB \).

11. To find the distance from \( A \) to \( P \) a distance, \( AB \), of 1000 yards is measured in a convenient direction. At \( A \) the angle \( PAB \) is found to be \( 41^\circ 18' \) and at \( B \) the angle \( PHA \) is found to be \( 114^\circ 38' \). What is the required distance to the nearest yard?
CHAPTER XIV.

HEIGHTS AND DISTANCES.

191. In the present chapter we shall consider some questions of the kind which occur in land-surveying. Simple questions of this kind have already been considered in Chapter III.

192. To find the height of an inaccessible tower by means of observations made at distant points.

Suppose $PQ$ to be the tower and that the ground passing through the foot $Q$ of the tower is horizontal. At a point $A$ on this ground measure the angle of elevation $\alpha$ of the top of the tower.

Measure off a distance $AB(=a)$ from $A$ directly toward the foot of the tower, and at $B$ measure the angle of elevation $\beta$.

To find the unknown height $x$ of the tower, we have to connect it with the measured length $a$. This is best done as follows:
From the triangle $PBQ$, we have

\[ \frac{x}{BP} = \sin \beta \quad \cdots \cdots \cdots \cdots \quad (1), \]

and, from the triangle $PAB$, we have

\[ \frac{PB}{a} = \frac{\sin PAB}{\sin BPA} = \frac{\sin \alpha}{\sin (\beta - \alpha)} \quad \cdots \cdots \cdots \cdots (2), \]

since $\angle BPA = \angle QBP - \angle QAP = \beta - \alpha$.

From (1) and (2), by multiplication, we have

\[ \frac{x}{a} = \frac{\sin \alpha \sin \beta}{\sin (\beta - \alpha)}, \]

i.e.

\[ x = a \frac{\sin \alpha \sin \beta}{\sin (\beta - \alpha)}. \]

The height $x$ is therefore given in a form suitable for logarithmic calculation.

**Numerical Example.** If $a = 100$ feet, $\alpha = 30^\circ$, and $\beta = 60^\circ$, then

\[ x = 100 \frac{\sin 30^\circ \sin 60^\circ}{\sin 30^\circ} = 100 \times \frac{\sqrt{3}}{2} = 86.6 \text{ feet.} \]

193. It is often not convenient to measure $AB$ directly towards $Q$.

Measure therefore $AB$ in any other suitable direction on the horizontal ground, and at $A$ measure the angle of elevation $\alpha$ of $P$, and also the angle $PAB (= \beta)$.

At $B$ measure the angle $PBA$ ($= \gamma$).

In the triangle $PAB$, we have then

\[ \angle APB = 180^\circ - \angle PAB - \angle PBA = 180^\circ - (\beta + \gamma). \]
Hence \[ \frac{AP}{a} = \frac{\sin PBA}{\sin BPA} = \frac{\sin \gamma}{\sin (\beta + \gamma)}. \]

From the triangle \( PAQ \), we have
\[ x = AP \sin \alpha = a \frac{\sin \alpha \sin \gamma}{\sin (\beta + \gamma)}. \]

Hence \( x \) is found by an expression suitable for logarithmic calculation.

194. To find the distance between two inaccessible points by means of observations made at two points the distance between which is known, all four points being supposed to be in one plane.

Let \( P \) and \( Q \) be two points whose distance apart, \( PQ \), is required.

Let \( A \) and \( B \) be the two known points whose distance apart, \( AB \), is given to be equal to \( a \).

At \( A \) measure the angles \( PAB \) and \( QAB \), and let them be \( \alpha \) and \( \beta \) respectively.

At \( B \) measure the angle \( PBA \) and \( QBA \), and let them be \( \gamma \) and \( \delta \) respectively.

Then in the triangle \( PAB \) we have one side \( a \) and the two adjacent angles \( \alpha \) and \( \gamma \) given, so that, as in Art. 163, we have \( AP \) given by the relation
\[ \frac{AP}{a} = \frac{\sin \gamma}{\sin APB} = \frac{\sin \gamma}{\sin (\alpha + \gamma)} \tag{1}. \]

In the triangle \( QAB \) we have, similarly,
\[ \frac{AQ}{a} = \frac{\sin \delta}{\sin (\beta + \delta)} \tag{2}. \]
In the triangle $\triangle APQ$ we have now determined the sides $AP$ and $AQ$; also the included angle $\angle PAQ (= \alpha - \beta)$ is known. We can therefore find the side $PQ$ by the method of Art. 181.

If the four points $A$, $B$, $P$, and $Q$ be not in the same plane, we must, in addition, measure the angle $\angle PAQ$; for in this case $\angle PAQ$ is not equal to $\alpha - \beta$. In other respects the solution will be the same as above.

195. **Bearings and Points of the Compass.** The Bearing of a given point $B$ as seen from a given point $O$ is the direction in which $B$ is seen from $O$. Thus if the direction of $OB$ bisect the angle between East and North, the bearing of $B$ is said to be North-East.

If a line is said to bear $20^\circ$ West of North, we mean that it is inclined to the North direction at an angle of $20^\circ$, this angle being measured from the North towards the West.
To facilitate the statement of the bearing of a point the circumference of the mariner's compass-card is divided into 32 equal portions, as in the above figure, and the subdivisions marked as indicated. Consider only the quadrant between East and North. The middle point of the arc between N. and E. is marked North-East (N.E.). The bisectors of the arcs between N.E. and N. and E. are respectively called North-North-East and East-North-East (N.N.E. and E.N.E.). The other four subdivisions, reckoning from N., are called North by East, N.E. by North, N.E. by East, and East by North. Similarly the other three quadrants are subdivided.

It is clear that the arc between two subdivisions of the card subtends an angle of \( \frac{360^\circ}{32} \), i.e. \( 11\frac{1}{4}^\circ \), at the centre \( O \).

**EXAMPLES. XXXIII.**

1. A flagstaff stands on the middle of a square tower. A man on the ground, opposite the middle of one face and distant from it 100 feet, just sees the flag; on his receding another 100 feet, the tangents of elevation of the top of the tower and the top of the flagstaff are found to be \( \frac{1}{2} \) and \( \frac{5}{9} \). Find the dimensions of the tower and the height of the flagstaff, the ground being horizontal.

2. A man, walking on a level plane towards a tower, observes that at a certain point the angular height of the tower is 10°, and, after going 50 yards nearer the tower, the elevation is found to be 15°. Having given

\[
L \sin 15^\circ = 9.4129962, \quad L \cos 5^\circ = 0.9983442,
\]

\[
\log 25.783 = 1.4113334, \quad \text{and} \quad \log 25.784 = 1.4113503,
\]

find, to 4 places of decimals, the height of the tower in yards.
3. \( DE \) is a tower standing on a horizontal plane and \( AI:CD \) is a straight line in the plane. The height of the tower subtends an angle \( \theta \) at \( A \), \( 2\theta \) at \( B \), and \( 3\theta \) at \( C \). If \( AB \) and \( BC \) be respectively 50 and 20 feet, find the height of the tower and the distance \( CD \).

4. A tower, 50 feet high, stands on the top of a mound; from a point on the ground the angles of elevation of the top and bottom of the tower are found to be \( 75^\circ \) and \( 45^\circ \) respectively; find the height of the mound.

5. A vertical pole (more than 100 feet high) consists of two parts, the lower being \( \frac{1}{3} \) of the whole. From a point in a horizontal plane through the foot of the pole and 40 feet from it, the upper part subtends an angle whose tangent is \( \frac{1}{2} \). Find the height of the pole.

6. A tower subtends an angle \( \alpha \) at a point on the same level as the foot of the tower, and at a second point, \( h \) feet above the first, the depression of the foot of the tower is \( \beta \). Find the height of the tower.

7. A person in a balloon, which has ascended vertically from flat land at the sea level, observes the angle of depression of a ship at anchor to be \( 30^\circ \); after descending vertically for 600 feet, he finds the angle of depression to be \( 15^\circ \); find the horizontal distance of the ship from the point of ascent.

8. \( PQ \) is a tower standing on a horizontal plane, \( Q \) being its foot; \( A \) and \( B \) are two points on the plane such that the \( \angle QAB \) is \( 90^\circ \), and \( AB \) is 40 feet. It is found that
\[
\cot \angle PAQ = \frac{3}{10} \quad \text{and} \quad \cot \angle PBQ = \frac{1}{2}.
\]
Find the height of the tower.

9. A column is E.S.E. of an observer, and at noon the end of the shadow is North-East of him. The shadow is 80 feet long and the elevation of the column at the observer's station is \( 45^\circ \). Find the height of the column.

10. A tower is observed from two stations \( A \) and \( B \). It is found to be due north of \( A \) and north-west of \( B \). \( B \) is due east of \( A \) and distant from it 100 feet. The elevation of the tower as seen from \( A \) is the complement of the elevation as seen from \( B \). Find the height of the tower.
11. The elevation of a steeple at a place due south of it is $45^\circ$ and at another place due west of the former place the elevation is $15^\circ$. If the distance between the two places be $a$, prove that the height of the steeple is

$$a \left( \sqrt{3} - 1 \right) \div 2 \sqrt{3}.$$

12. A person stands in the diagonal produced of the square base of a church tower, at a distance $2a$ from it, and observes the angles of elevation of each of the two outer corners of the top of the tower to be $30^\circ$, whilst that of the nearest corner is $45^\circ$. Prove that the breadth of the tower is $a \left( \sqrt{10} - \sqrt{2} \right)$.

13. A person standing at a point $A$ due south of a tower built on a horizontal plane observes the altitude of the tower to be $60^\circ$. He then walks to $B$ due west of $A$ and observes the altitude to be $45^\circ$, and again at $C$ in $AB$ produced he observes it to be $30^\circ$. Prove that $B$ is midway between $A$ and $C$.

14. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is $\theta$ and that at the middle point it is $\phi$. Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta) \sin (\phi - \theta)}}.$$

15. $A$ and $B$ are two stations 1000 feet apart; $P$ and $Q$ are two stations in the same plane as $AB$ and on the same side of it; the angles $PAB$, $PBA$, $QAB$, and $QBA$ are respectively $75^\circ$, $30^\circ$, $45^\circ$, and $90^\circ$; find how far $P$ is from $Q$ and how far each is from $A$ and $B$.

For the following seven examples a book of tables will be wanted.

16. At a point on a horizontal plane the elevation of the summit of a mountain is found to be $22^\circ 15'$, and at another point on the plane, a mile further away in a direct line, its elevation is $10^\circ 12'$; find the height of the mountain.

17. From the top of a hill the angles of depression of two successive milestones, on level ground and in the same vertical plane with the observer, are found to be $5^\circ$ and $10^\circ$ respectively. Find the height of the hill and the horizontal distance to the nearest milestone.

18. A castle and a monument stand on the same horizontal plane. The height of the castle is 140 feet, and the angles of depression of the top and bottom of the monument as seen from the top of the castle are $40^\circ$ and $80^\circ$ respectively. Find the height of the monument.
19. A flagstaff \( PN \) stands on level ground. A base \( AB \) is measured at right angles to \( AN \), the points \( A, B, \) and \( N \) being in the same horizontal plane, and the angles \( PAN \) and \( PBN \) are found to be \( \alpha \) and \( \beta \) respectively. Prove that the height of the flagstaff is

\[
AB = \frac{\sin \alpha \sin \beta}{\sqrt{\sin (\alpha - \beta) \sin (\alpha + \beta)}}.
\]

If \( AB = 100 \) feet, \( \alpha = 70^\circ \), and \( \beta = 50^\circ \), calculate the height.

20. A man, standing due south of a tower on a horizontal plane through its foot, finds the elevation of the top of the tower to be \( 54^\circ 16' \); he goes east 100 yards and finds the elevation to be then \( 50^\circ 8' \). Find the height of the tower.

21. A man in a balloon observes that the angle of depression of an object on the ground bearing due north is \( 33^\circ \); the balloon drifts 3 miles due west and the angle of depression is now found to be \( 21^\circ \). Find the height of the balloon.

22. From the extremities of a horizontal base-line \( AB \), whose length is \( 1000 \) feet, the bearings of the foot \( C \) of a tower are observed and it is found that \( \angle CAB = 56^\circ 23' \), \( \angle CBA = 47^\circ 15' \), and that the elevation of the tower from \( A \) is \( 9^\circ 25' \); find the height of the tower.

196. Ex. 1. A flagstaff is on the top of a tower which stands on a horizontal plane. A person observes the angles, \( \alpha \) and \( \beta \), subtended at a point on the horizontal plane by the flagstaff and the tower; he then walks a known distance \( a \) toward the tower and finds that the flagstaff subtends the same angle as before; prove that the height of the tower and the length of the flagstaff are respectively

\[
\frac{a \sin \beta \cos (\alpha + \beta)}{\cos (\alpha + 2\beta)} \quad \text{and} \quad \frac{a \sin \alpha}{\cos (\alpha + 2\beta)}.
\]

Let \( P \) and \( Q \) be the top and foot of the tower, and let \( PR \) be the flagstaff. Let \( A \) and \( B \) be the points at which the measurements are taken, so that \( \angle PAQ = \beta \) and \( \angle PAR = \angle PBR = \alpha \). Since the two latter angles are equal, a circle will go through the four points \( A, B, P, \) and \( R \).
To find the height of the flagstaff we have to connect the unknown length $PR$ with the known length $AB$.

This may be done by connecting each with the length $AR$.

To do this, we must first determine the angles of the triangles $ARP$ and $ARB$.

Since $A$, $B$, $P$, and $R$ lie on a circle, we have

$$\angle BRP = \angle BAP = \beta,$$

and

$$\angle APB = \angle ARB = \theta \text{ (say)}.$$

Also

$$\angle APR = 90^\circ + \angle PAQ = 90^\circ + \beta$$

Hence, since the angles of the triangle $APR$ are together equal to two right angles, we have

$$180^\circ = \alpha + (90^\circ + \beta) + (\theta + \beta),$$

so that

$$\theta = 90^\circ - (\alpha + 2\beta) \quad \text{(1)}.$$

From the triangles $APR$ and $ABR$ we then have

$$\frac{PR}{\sin \alpha} = \frac{AR}{\sin RPA} = \frac{AR}{\sin RBA} = \frac{a}{\sin \theta} \quad \text{(Art. 163)}.$$

[It will be found in Chap. XV. that each of these quantities is equal to the diameter of the circle.]

Hence the height of the flagstaff

$$= PR = \frac{a \sin \alpha}{\sin \theta} = \frac{a \sin \alpha}{\cos (\alpha + 2\beta)}, \text{ by (1)}.$$

Again,

$$\frac{PQ}{PB} = \cos BPQ = \cos (\alpha + \beta) \quad \text{(2)},$$

and

$$\frac{PB}{a} = \frac{\sin PAB}{\sin APB} = \frac{\sin \beta}{\sin \theta} \quad \text{(3)}.$$

Hence, from (2) and (3), by multiplication,

$$\frac{PQ}{a} = \frac{\sin \beta \cos (\alpha + \beta)}{\sin \theta} = \frac{\sin \beta \cos (\alpha + \beta)}{\cos (\alpha + 2\beta)}, \text{ by (1)}. $$
Also, $BQ = PQ \tan B\!P\!Q = PQ \tan (\alpha + \beta)$
\[ = a \frac{\sin \beta \sin (\alpha + \beta)}{\cos (\alpha + 2\beta)}, \]
and $A\!Q = a + BQ = a \frac{\cos (\alpha + 2\beta) + \sin \beta \sin (\alpha + \beta)}{\cos (\alpha + 2\beta)}$
\[ = a \frac{\cos \beta \cos (\alpha + \beta)}{\cos (\alpha + 2\beta)}. \]

If $a$, $\alpha$, and $\beta$ be given numerically, these results are all in a form suitable for logarithmic computation.

**Ex. 2.** At a distance $a$ from the foot of a tower $AB$, of known height $b$, a flagstaff $BC$ and the tower subtend equal angles. Find the height of the flagstaff.

Let $O$ be the point of observation, and let the angles $AOB$ and $BOC$ be each $\theta$; also let the height $BC$ be $y$.

We then have $\tan \theta = \frac{b}{a}$, and $\tan 2\theta = \frac{b + y}{a}$.

Hence
\[ \frac{b + y}{a} = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \frac{b}{a}}{1 - \left(\frac{b}{a}\right)^2}, \]
so that
\[ \frac{b + y}{a} = \frac{2ab}{a^2 - b^2}. \]

Then
\[ y = \frac{2a^2b}{a^2 - b^2} - b = b \frac{a^2 + b^2}{a^2 - b^2}. \]

If $a$ and $b$ be given numerically, we thus easily obtain $y$.

**197. Ex.** A man walks along a straight road and observes that the greatest angle subtended by two objects is $\alpha$; from the point where this greatest angle is subtended he walks a distance $c$ along the road, and finds that the two objects are now in a straight line which makes an angle $\beta$ with the road; prove that the distance between the objects is
\[ c \sin \alpha \sin \beta \sec \frac{\alpha + \beta}{2} \sec \frac{\alpha - \beta}{2}. \]
Let $P$ and $Q$ be the two points, and let $PQ$ meet the road in $B$.

If $A$ be the point at which the greatest angle is subtended, then $A$ must be the point where a circle drawn through $P$ and $Q$ touches the road.

[For, take any other point $A'$ on $AB$, and join it to $P$ cutting the circle in $B'$, and join $A'Q$ and $B'Q$.

Then $\angle PA'Q < \angle PB'Q$ (Euc. I. 16), and therefore $\angle PAQ < \angle PAQ$ (Euc. III. 21).]

Let the angle $QAB$ be called $\theta$. Then (Euc. III. 32) the angle $APQ$ is $\theta$ also.

Hence $180^\circ = \text{sum of the angles of the triangle } PAB$

$$= \theta + (\alpha + \theta) + \beta,$$

so that $\theta = 90^\circ - \frac{\alpha + \beta}{2}$.

From the triangles $PAQ$ and $QAB$ we have

$$\frac{PQ}{AQ} = \frac{\sin \alpha}{\sin \theta}, \quad \text{and} \quad \frac{AQ}{\alpha} = \frac{\sin \beta}{\sin AQB} = \frac{\sin \beta}{\sin (\theta + \alpha)}.$$
Hence, by multiplication, we have

\[
\frac{PQ}{\sigma} = \frac{\sin \alpha \sin \beta}{\sin \theta \sin (\theta + \alpha)} = \frac{\sin \alpha \sin \beta}{\cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}}.
\]

\[
\therefore PQ = c \sin \alpha \sin \beta \sec \frac{\alpha + \beta}{2} \sec \frac{\alpha - \beta}{2}.
\]

**EXAMPLES. XXXIV.**

1. A bridge has 5 equal spans, each of 100 feet measured from the centre of the piers, and a boat is moored in a line with one of the middle piers. The whole length of the bridge subtends a right angle as seen from the boat. Prove that the distance of the boat from the bridge is 100\(\sqrt{6}\) feet.

2. A ladder placed at an angle of 75° with the ground just reaches the sill of a window at a height of 27 feet above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street it is at an angle of 15° with the ground. Prove that the breadth of the street and the length of the ladder are respectively

\[
27(3 - \sqrt{3}) \text{ and } 27(\sqrt{6} - \sqrt{2}) \text{ feet.}
\]

3. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house; from the level of the street the angle subtended is the angle whose tangent is 3; from two windows one above the other the angle subtended is found to be the angle whose tangent is -3; the height of the opposite house being 60 feet, find the height above the street of each of the two windows.

4. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground; find the longest shadow it can cast on the ground.

Calculate the altitude of the sun when the longest shadow it can cast is 3\(\frac{1}{2}\) times the length of the rod.

5. A person on a ship \(A\) observes another ship \(B\) leaving a harbour, whose bearing is then N.W. After 10 minutes \(A\), having sailed one mile
N.E., sees $B$ due west and the harbour then bears $60^\circ$ West of North. After another 10 minutes $B$ is observed to bear S.W. Find the distances between $A$ and $B$ at the first observation and also the direction and rate of $B$.

6. A person on a ship sailing north sees two lighthouses, which are 6 miles apart, in a line due west; after an hour's sailing one of them bears S.W. and the other S.S.W. Find the ship's rate.

7. A person on a ship sees a lighthouse N.W. of himself. After sailing for 12 miles in a direction $15^\circ$ south of W. the lighthouse is seen due N. Find the distance of the lighthouse from the ship in each position.

8. A man, travelling west along a straight road, observes that when he is due south of a certain windmill the straight line drawn to a distant tower makes an angle of $30^\circ$ with the road. A mile further on the bearings of the windmill and tower are respectively N.E. and N.W. Find the distances of the tower from the windmill and from the nearest point of the road.

9. An observer on a headland sees a ship due north of him; after a quarter of an hour he sees it due east and after another half-hour he sees it due south-east; find the direction that the ship's course makes with the meridian and the time after the ship is first seen until it is nearest the observer, supposing that it sails uniformly in a straight line.

10. A man walking along a straight road, which runs in a direction $30^\circ$ east of north, notes when he is due south of a certain house; when he has walked a mile further, he observes that the house lies due west and that a windmill on the opposite side of the road is N.E. of him; three miles further on he finds that he is due north of the windmill; prove that the line joining the house and the windmill makes with the road the angle whose tangent is $\frac{48 - 25\sqrt{3}}{11}$.

11. $A$, $B$, and $C$ are three consecutive milestones on a straight road from each of which a distant spire is visible. The spire is observed to bear north-east at $A$, east at $B$, and $60^\circ$ east of south at $C$. Prove that the shortest distance of the spire from the road is $\frac{7 + 5\sqrt{3}}{13}$ miles.

12. Two stations due south of a tower, which leans towards the north, are at distances $a$ and $b$ from its foot; if $a$ and $b$ be the
elevations of the top of the tower from these stations, prove that its inclination to the horizontal is

$$\cot^{-1} \frac{b \cot a - a \cot \beta}{b - a}.$$

13. From a point $A$ on a level plane the angle of elevation of a balloon is $a$, the balloon being south of $A$; from a point $B$, which is at a distance $c$ south of $A$, the balloon is seen northwards at an elevation of $\beta$; find the distance of the balloon from $A$ and its height above the ground.

14. A statue on the top of a pillar subtends the same angle $a$ at distances of 9 and 11 yards from the pillar; if $\tan a = \frac{1}{10}$, find the height of the pillar and of the statue.

15. A flagstaff on the top of a tower is observed to subtend the same angle $a$ at two points on a horizontal plane, which lie on a line passing through the centre of the base of the tower and whose distance from one another is $2a$, and an angle $\beta$ at a point halfway between them. Prove that the height of the flagstaff is

$$a \sin a \sqrt{\frac{2 \sin \beta}{\cos a \sin (\beta - a)}}.$$

16. An observer in the first place stations himself at a distance $a$ feet from a column standing upon a mound. He finds that the column subtends an angle, whose tangent is $\frac{1}{2}$, at his eye which may be supposed to be on the horizontal plane through the base of the mound. On moving $\frac{2}{3}a$ feet nearer the column, he finds that the angle subtended is unchanged. Find the height of the mound and of the column.

17. A church tower stands on the bank of a river, which is 150 feet wide, and on the top of the tower is a spire 30 feet high. To an observer on the opposite bank of the river, the spire subtends the same angle that a pole six feet high subtends when placed upright on the ground at the foot of the tower. Prove that the height of the tower is nearly 285 feet.

18. A person, wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top is $30^\circ$. On walking a distance $a$ in a certain direction he finds that the elevation of the top is the same as before, and on then walking a distance $\frac{5}{3}a$ at right angles to his former direction he finds the
elevation of the top to be 60°. Prove that the height of the tower is either \( \sqrt{\frac{5}{6}} a \) or \( \sqrt{\frac{85}{48}} a \).

19. The angles of elevation of the top of a tower, standing on a horizontal plane, from two points distant \( a \) and \( b \) from the base and in the same straight line with it are complementary. Prove that the height of the tower is \( \sqrt{ab} \) feet, and, if \( \theta \) be the angle subtended at the top of the tower by the line joining the two points, then \( \sin \theta = \frac{a - b}{a + b} \).

20. A tower 150 feet high stands on the top of a cliff 80 feet high. At what point on the plane passing through the foot of the cliff must an observer place himself so that the tower and the cliff may subtend equal angles, the height of his eye being 5 feet?

21. A statue on the top of a pillar, standing on level ground, is found to subtend the greatest angle at the eye of an observer when his distance from the pillar is \( c \) feet; prove that the height of the statue is \( 2c \tan a \) feet, and find the height of the pillar.

22. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°. A line 100 feet in length was measured straight up the incline from the foot of the tower, and at the end of this line the tower subtended an angle of 54°. Find the height of the tower, having given \( \log 2 = 0.30103 \), \( \log 114.4123 = 2.0584726 \), and \( L \sin 54° = 0.9079576 \).

23. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet, and then finds that the tower subtends an angle of 30°. Prove that the height of the tower is 40 (\( \sqrt{6} - \sqrt{2} \)) feet.

24. The altitude of a certain rock is 47°, and after walking towards it 1000 feet up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the vertical height of the rock above the first point of observation, given that \( \sin 47° = 0.73135 \).

25. A man observes that when he has walked \( c \) feet up an inclined plane the angular depression of an object in a horizontal plane through the foot of the slope is \( a \), and that, when he has walked a further distance of \( c \) feet, the depression is \( \beta \). Prove that the inclination of the slope to the horizon is the angle whose cotangent is

\[
(2 \cot \beta - \cot a).
\]

L. T.
26. A regular pyramid on a square base has an edge 150 feet long, and the length of the side of its base is 200 feet. Find the inclination of its face to the base.

27. A pyramid has for base a square of side $a$; its vertex lies on a line through the middle point of the base and perpendicular to it, and at a distance $h$ from it; prove that the angle $a$ between the two lateral faces is given by the equation

$$\sin a = \frac{2h \sqrt{2a^2 + 4h^3}}{a^2 + 4h^2}.$$ 

28. A flagstaff, 100 feet high, stands in the centre of an equilateral triangle which is horizontal. From the top of the flagstaff each side subtends an angle of $60^\circ$; prove that the length of the side of the triangle is $50\sqrt{6}$ feet.

29. The extremity of the shadow of a flagstaff, which is 6 feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant 56 and 8 feet respectively from the extremities of that side. Find the sun’s altitude if the height of the pyramid be 34 feet.

30. The extremity of the shadow of a flagstaff, which is 6 feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant $x$ feet and $y$ feet respectively from the ends of that side; prove that the height of the pyramid is

$$\sqrt{x^2 + y^2} \tan a - 6,$$

where $a$ is the elevation of the sun.

31. The angle of elevation of a cloud from a point $h$ feet above a lake is $a$, and the angle of depression of its reflection in the lake is $\beta$; prove that its height is $h \frac{\sin (\beta + a)}{\sin (\beta - a)}$.

32. The shadow of a tower is observed to be half the known height of the tower and sometime afterwards it is equal to the known height; how much will the sun have gone down in the interval, given

$$\log 2 = 0.30103, \quad L \tan 63^\circ 26' = 10.3099994,$$

and diff. for $1' = 3159$?
33. An isosceles triangle of wood is placed in a vertical plane, vertex upwards, and faces the sun. If $2a$ be the base of the triangle, $h$ its height, and $30^\circ$ the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is $\frac{2ah\sqrt{3}}{3h^2 - a^2}$.

34. A rectangular target faces due south, being vertical and standing on a horizontal plane. Compare the area of the target with that of its shadow on the ground when the sun is $\beta^\circ$ from the south at an altitude of $\alpha^\circ$.

35. A spherical ball, of diameter $\delta$, subtends an angle $\alpha$ at a man's eye when the elevation of its centre is $\beta$; prove that the height of the centre of the ball is $\frac{1}{2}\delta \sin \beta \csc \frac{\alpha}{2}$.

36. A man standing on a plane observes a row of equal and equidistant pillars, the 10th and 17th of which subtend the same angle that they would do if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height. Prove that, neglecting the height of the man's eye, the line of pillars is inclined to the line drawn from his eye to the first at an angle whose secant is nearly 2·6.

For the following four examples a book of tables will be wanted.

37. $A$ and $B$ are two points, which are on the banks of a river and opposite to one another, and between them is the mast, $PN$, of a ship; the breadth of the river is 1000 feet, and the angular elevation of $P$ at $A$ is $14^\circ 20'$ and at $B$ it is $8^\circ 10'$. What is the height of $P$ above $AB$?

38. $AB$ is a line 1000 yards long; $B$ is due north of $A$ and from $B$ a distant point $P$ bears $70^\circ$ east of north; at $A$ it bears $41^\circ 22'$ east of north; find the distance from $A$ to $P$.

39. $A$ is a station exactly 10 miles west of $B$. The bearing of a particular rock from $A$ is $74^\circ 19'$ east of north, and its bearing from $B$ is $26^\circ 51'$ west of north. How far is it north of the line $AB$?

40. The summit of a spire is vertically over the middle point of a horizontal square enclosure whose side is of length $a$ feet; the height of the spire is $h$ feet above the level of the square. If the shadow of the spire just reach a corner of the square when the sun has an altitude $\theta$, prove that

$$h\sqrt{2}=a\tan\theta.$$ 

Calculate $h$, having given $a=1000$ feet and $\theta=25^\circ 15'$. 

15—2
CHAPTER XV.

PROPERTIES OF A TRIANGLE.

198. Area of a given triangle. Let $ABC$ be any triangle, and $AD$ the perpendicular drawn from $A$ upon the opposite side.

Through $A$ draw $EAF$ parallel to $BC$, and draw $BE$ and $CF$ perpendicular to it. By Eucl. I. 41, the area of the triangle $ABC$

$$= \frac{1}{2} \text{rectangle } BF = \frac{1}{2} BC \cdot CF = \frac{1}{2} a \cdot AD.$$

But $AD = AB \sin B = c \sin B$.

The area of the triangle $ABC$ therefore $= \frac{1}{2} ca \sin B$.

This area is denoted by $\Delta$.

Hence $\Delta = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A \ldots (1)$.

By Art. 169, we have $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$, so that $\Delta = \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)} \ldots (2)$.

This latter quantity is often called $S$. 
EXAMPLES. XXXV.

Find the area of the triangle $ABC$ when

1. $a=13$, $b=14$, and $c=15$.  
2. $a=18$, $b=24$, and $c=30$.
3. $a=25$, $b=52$, and $c=63$.  
4. $a=125$, $b=123$, and $c=62$.
5. $a=15$, $b=36$, and $c=39$.  
6. $a=287$, $b=810$, and $c=865$.
7. $a=35$, $b=84$, and $c=91$.

8. $a=\sqrt{3}$, $b=\sqrt{2}$, and $c=\frac{\sqrt{6}+\sqrt{2}}{2}$.

9. If $B=45^\circ$, $C=60^\circ$, and $a=2(\sqrt{3}+1)$ inches, prove that the area of the triangle is $6+2\sqrt{3}$ sq. inches.

10. The sides of a triangle are 119, 111, and 92 yards; prove that its area is 10 sq. yards less than an acre.

11. The sides of a triangular field are 242, 1212, and 1450 yards; prove that the area of the field is 6 acres.

12. A workman is told to make a triangular enclosure of sides 50, 41, and 21 yards respectively; having made the first side one yard too long, what length must he make the other two sides in order to enclose the prescribed area with the prescribed length of fencing?

13. Find, correct to .0001 of an inch, the length of one of the equal sides of an isosceles triangle on a base of 14 inches having the same area as a triangle whose sides are 13.6, 15, and 15.4 inches.

14. Prove that the area of a triangle is $\frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$.

If one angle of a triangle be 60°, the area 10$\sqrt{3}$ square feet, and the perimeter 20 feet, find the lengths of the sides.

15. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ths of an equilateral triangle of the same perimeter; prove that its sides are in the ratio 3:5:7, and find the greatest angle of the triangle.

16. In a triangle the least angle is 45° and the tangents of the angles are in A.P. If its area be 3 square yards, prove that the lengths of the sides are $3\sqrt{5}$, $6\sqrt{2}$, and 9 feet, and that the tangents of the other angles are respectively 2 and 3.
17. The lengths of two sides of a triangle are one foot and \( \sqrt{2} \) feet respectively, and the angle opposite the shorter side is 30°; prove that there are two triangles satisfying these conditions, find their angles, and shew that their areas are in the ratio

\[ \sqrt{3} + 1 : \sqrt{3} - 1. \]

18. Find by the aid of the tables the area of the larger of the two triangles given by the data

\[ A = 31^\circ 15', \ a = 5 \text{ ins.}, \text{ and } b = 7 \text{ ins}. \]

199. On the circles connected with a given triangle.

The circle which passes through the angular points of a triangle \( ABC \) is called its circumscribing circle or, more briefly, its circumcircle. The centre of this circle is found by the construction of Euc. iv. 5. Its radius is always called \( R \).

The circle which can be inscribed within the triangle so as to touch each of the sides is called its inscribed circle or, more briefly, its incircle. The centre of this circle is found by the construction of Euc. iv. 4. Its radius will be denoted by \( r \).

The circle which touches the side \( BC \) and the two sides \( AB \) and \( AC \) produced is called the escribed circle opposite the angle \( A \). Its radius will be denoted by \( r_1 \).

Similarly \( r_2 \) denotes the radius of the circle which touches the side \( CA \) and the two sides \( BC \) and \( BA \) produced. Also \( r_3 \) denotes the radius of the circle touching \( AB \) and the two sides \( CA \) and \( CB \) produced.

200. To find the magnitude of \( R \), the radius of the circumcircle of any triangle \( ABC \).

Bisect the two sides \( BC \) and \( CA \) in \( D \) and \( E \) respectively, and draw \( DO \) and \( EO \) perpendicular to \( BC \) and \( CA \).
By Eucl. iv. 5, $O$ is the centre of the circumcircle. Join $OB$ and $OC$.

The point $O$ may either lie within the triangle as in Fig. 1, or without it as in Fig. 2, or upon one of the sides as in Fig. 3.

Taking the first figure, the two triangles $BOD$ and $COD$ are equal in all respects, so that

$$\angle BOD = \angle COD,$$

$$\therefore \angle BOD = \frac{1}{2} \angle BOC = \angle BAC \quad \text{(Eucl. iii. 20)},$$

$$= A.$$

Also $BD = BO \sin BOD$.

$$\therefore \frac{a}{2} = R \sin A.$$

If $A$ be obtuse, as in Fig. 2, we have

$$\angle BOD = \frac{1}{2} \angle BOC = \angle BLC = 180^\circ - A \quad \text{(Eucl. iii. 22)},$$

so that, as before, $\sin BOD = \sin A$,

and

$$R = \frac{a}{2 \sin A}.$$

If $A$ be a right angle, as in Fig. 3, we have

$$R = OA = OC = \frac{a}{2}$$

$$= \frac{a}{2 \sin A},$$ since in this case $\sin A = 1$. 

The relation found above is therefore true for all triangles.
Hence, in all three cases, we have

\[ R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} \quad \text{(Art. 163)} \]

201. In Art. 169 we have shown that

\[ \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2S}{bc}, \]

where \( S \) is the area of the triangle.
Substituting this value of \( \sin A \) in (1), we have

\[ R = \frac{abc}{4S}, \]

giving the radius of the circumcircle in terms of the sides.

202. To find the value of \( r \), the radius of the incircle of the triangle \( ABC \).

Bisect the two angles \( B \) and \( C \) by the two lines \( BI \) and \( CI \) meeting in \( I \).

By Eucl. iii. 4, \( I \) is the centre of the incircle. Join \( IA \), and draw \( ID, IE, \) and \( IF \) perpendicular to the three sides.

Then \( ID = IE = IF = r \).
We have

\[ \text{area of } \triangle IBC = \frac{1}{2} ID \cdot BC = \frac{1}{2} r \cdot a, \]
\[ \text{area of } \triangle ICA = \frac{1}{2} IE \cdot CA = \frac{1}{2} r \cdot b, \]
and
\[ \text{area of } \triangle IAB = \frac{1}{2} IF \cdot AB = \frac{1}{2} r \cdot c. \]
Hence, by addition, we have
\[ \frac{1}{2}r \cdot a + \frac{1}{2}r \cdot b + \frac{1}{2}r \cdot c = \text{sum of the areas of the triangles } IBC, ICA, \text{ and } IAB \]
\[ = \text{area of the } \Delta ABC, \]
i.e.
\[ r \frac{a + b + c}{2} = S, \]
so that
\[ r \cdot s = S. \]
\[ \therefore r = \frac{S}{s}. \]

203. Since the angles \( IBD \) and \( IDB \) are respectively equal to the angles \( IBF \) and \( IFB \), the two triangles \( IDB \) and \( IFB \) are equal in all respects.

Hence \( BD = BF \), so that \( 2BD = BD + BF \).

So also \( AE = AF \), so that \( 2AE = AE + AF \),
and \( CE = CD \), so that \( 2CE = CE + CD \).

Hence, by addition, we have
\[ 2BD + 2CE = (BD + CD) + (CE + AE) + (AF + FB), \]
i.e.
\[ 2BD + 2AC = BC + CA + AB. \]
\[ \therefore 2BD + 2b = a + b + c = 2s. \]

Hence \( BD = s - b = BF \);
so \( CE = s - c = CD \),
and \( AF = s - a = AE \).

Now \( \frac{ID}{BD} = \tan IBD = \tan \frac{B}{2} \).
\[ \therefore r = ID = BD \tan \frac{B}{2} = (s - b) \tan \frac{B}{2}. \]
So \[ r = IE = CE \tan ICE = (s - c) \tan \frac{C}{2}, \]
and also \[ r = IF = FA \tan IAF = (s - a) \tan \frac{A}{2}. \]

Hence \[ r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}. \]

204. A third value for \( r \) may be found as follows: we have \[ a = BD + DC = ID \cot IBD + ID \cot IDC \]

\[ = r \cot \frac{B}{2} + r \cot \frac{C}{2} \]

\[ = r \left[ \frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \right]. \]

\[ \therefore a \sin \frac{B}{2} \sin \frac{C}{2} = r \left[ \sin \frac{C}{2} \cos \frac{B}{2} + \cos \frac{C}{2} \sin \frac{B}{2} \right] \]

\[ = r \sin \left( \frac{B}{2} + \frac{C}{2} \right) = r \sin \left[ 90^\circ - \frac{A}{2} \right] = r \cos \frac{A}{2}. \]

\[ \therefore r = a \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}. \]

**Cor.** Since \[ a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2}, \]

we have \[ r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \]

205. To find the value of \( r_1 \), the radius of the escribed circle opposite the angle \( A \) of the triangle \( ABC \).
Produce $AB$ and $AC$ to $L$ and $M$.

Bisect the angles $CBL$ and $BCM$ by the lines $BI_1$ and $CI_1$, and let these lines meet in $I_1$.

Draw $I_1D_1$, $I_1E_1$, and $I_1F_1$ perpendicular to the three sides respectively.

The two triangles $I_1D_1B$ and $I_1F_1B$ are equal in all respects, so that $I_1F_1 = I_1D_1$.

Similarly $I_1E_1 = I_1D_1$.

The three perpendiculars $I_1D_1$, $I_1E_1$, and $I_1F_1$ being equal, the point $I_1$ is the centre of the required circle.

Now the area $ABI_1C$ is equal to the sum of the triangles $ABC$ and $I_1BC$; it is also equal to the sum of the triangles $I_1BA$ and $I_1CA$.

Hence

\[ \Delta ABC + \Delta I_1BC = \Delta I_1CA + \Delta I_1AB. \]

\[ \therefore S + \frac{1}{2} I_1D_1 \cdot BC = \frac{1}{2} I_1E_1 \cdot CA + \frac{1}{2} I_1F_1 \cdot AB, \]

i.e. \[ S + \frac{1}{2} r_1 \cdot a = \frac{1}{2} r_1 \cdot b + \frac{1}{2} r_1 \cdot c. \]

\[ \therefore S = r_1 \left[ \frac{b + c - a}{2} \right] = r_1 \left[ \frac{b + c + a}{2} - a \right] = r_1 (s - a). \]

\[ \therefore r_1 = \frac{S}{s - a}. \]

Similarly it can be shewn that

\[ r_2 = \frac{S}{s - b}, \text{ and } r_3 = \frac{S}{s - c}. \]
206. Since $AE_1$ and $AF_1$ are tangents, we have, as in Art. 203, $AE_1 = AF_1$.

Similarly, $BF_1 = BD_1$, and $CE_1 = CD_1$.

$2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$

$= AB + BD_1 + AC + CD_1 = AB + BC + CA = 2s$.

$\therefore AE_1 = s = AF_1$.

Also, $BD_1 = BF_1 = AF_1 - AB = s - c$,

and $CD_1 = CE_1 = AE_1 - AC = s - b$.

$\therefore I_1E_1 = AE_1 \tan I_1AE_1$.

i.e. $r_1 = s \tan \frac{A}{2}$.

207. A third value may be obtained for $r_1$ in terms of $a$ and the angles $B$ and $C$.

For, since $I_1C$ bisects the angle $BCE_1$, we have

$\angle I_1CD_1 = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{C}{2}$.

So $\angle I_1BD_1 = 90^\circ - \frac{B}{2}$.

$\therefore a = BC = BD_1 + D_1C$

$= I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$

$= r_1 \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right)$

$= r_1 \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right)$.
\[ \therefore a \cos \frac{B}{2} \cos \frac{C}{2} = r_1 \left( \sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2} \right) \]

\[ = r_1 \sin \left( \frac{B}{2} + \frac{C}{2} \right) = r_1 \sin \left( 90^\circ - \frac{A}{2} \right) = r_1 \cos \frac{A}{2}. \]

\[ \therefore r_1 = a \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}. \]

\textbf{Cor.} Since \( a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2} \), we have

\[ r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \]

\textbf{EXAMPLES. XXXVI.}

1. In a triangle whose sides are 18, 24, and 30 inches respectively, prove that the circumradius, the inradius, and the radii of the three escribed circles are respectively 15, 6, 12, 18, and 30 inches.

2. The sides of a triangle are 13, 14, and 15 feet; prove that
   (1) \( R = \frac{87}{5} \) ft., (2) \( r = 4 \) ft., (3) \( r_1 = \frac{102}{5} \) ft.,
   (4) \( r_2 = 12 \) ft., and (5) \( r_3 = 11 \) ft.

3. In a triangle \( ABC \) if \( a = 13, b = 4, \) and \( \cos C = -\frac{5}{13} \), find \( R, r, r_1, r_2, \) and \( r_3. \)

4. In the ambiguous case of the solution of triangles prove that the circumcircles of the two triangles are equal.

Prove that

5. \( r_1 (s - a) = r_2 (s - b) = r_3 (s - c) = rs = S. \)

6. \( \frac{rr_1}{r_2r_3} = \tan^2 \frac{A}{2}. \)

7. \( rr_1r_2r_3 = S^2. \)

8. \( r_1r_2r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}. \)

9. \( rr_1 \cot \frac{A}{2} = S. \)
10. $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$.

11. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.

12. $a(r r_1 + r_2 r_3) = b(r r_2 + r_3 r_1) = c(r r_3 + r_1 r_2)$.

13. $(r_1 + r_2) \tan \frac{C}{2} = (r_3 - r) \cot \frac{C}{2} = c$.

14. $S = 2R^2 \sin A \sin B \sin C$.

15. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$.

16. $S = 4Rr \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

17. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{S^2}$.

18. $r_1 + r_2 + r_3 - r = 4R$.

19. $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

20. $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2lr}$.

21. $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

22. $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

208. Orthocentre and pedal triangle of any triangle.

Let $ABC$ be any triangle, and let $AK, BL,$ and $CM$ be the perpendiculars from $A, B,$ and $C$ upon the opposite sides of the triangle. It can be easily shewn, as in most books on Geometry, that these three perpendiculars meet in a common point $P$. This point $P$ is called the orthocentre of the triangle. The triangle $KLM$, which is formed by joining the feet of these perpendiculars, is called the pedal triangle of $ABC$.

209. Distances of the orthocentre from the angular points of the triangle.
We have \( PK = KB \tan \angle BPK = KB \tan (90^\circ - C) \)
\[
= AB \cos B \cot C = \frac{c}{\sin C} \cos B \cos C
\]
\[
= 2R \cos B \cos C \quad (\text{Art. 200}).
\]
Again \( AP = AL \cdot \sec KAC \)
\[
= c \cos A \cdot \cos C
\]
\[
= \frac{c}{\sin C} \cdot \cos A
\]
\[
= 2R \cos A \quad (\text{Art. 200}).
\]
So \( BP = 2R \cos B \), and \( CP = 2R \cos C \).

The distances of the orthocentre from the angular points are therefore \( 2R \cos A \), \( 2R \cos B \), and \( 2R \cos C \); its distances from the sides are \( 2R \cos B \cos C \), \( 2R \cos C \cos A \), and \( 2R \cos A \cos B \).

210. To find the sides and angles of the pedal triangle.

Since the angles \( PKC \) and \( PLC \) are right angles, the points \( P, L, C, \) and \( K \) lie on a circle.

\[
\therefore \angle PKL = \angle PCL \quad (\text{Euc. III. 21})
\]
\[
= 90^\circ - A.
\]

Similarly, \( P, K, B \) and \( M \) lie on a circle, and therefore
\[
\angle PKM = \angle PBM
\]
\[
= 90^\circ - A.
\]

Hence \( \angle MKL = 180^\circ - 2A \)
\[
= \text{the supplement of } 2A.
\]

So \( \angle KLM = 180^\circ - 2B \),
and \( \angle LMK = 180^\circ - 2C \).
Again, from the triangle $ALM$, we have

$$\frac{LM}{\sin A} = \frac{AL}{\sin AML} = \frac{AB \cos A}{\cos PAL} = \frac{c \cos A}{\cos PAL} = \frac{c \cos A}{\sin C}.$$  

$$\therefore \quad LM = \frac{c}{\sin C} \sin A \cos A$$

$$= a \cos A. \quad \text{(Art. 163.)}$$

So  

$$MK = b \cos B, \quad \text{and} \quad KL = c \cos C.$$  

The sides of the pedal triangle are therefore $a \cos A$, $b \cos B$, and $c \cos C$; also its angles are the supplements of twice the angles of the triangle.

211. Let $I$ be the centre of the incircle and $I_1$, $I_2$, and $I_3$ the centres of the escribed circles which are opposite to $A$, $B$, and $C$ respectively. As in Arts. 202 and 205, $IC$ bisects the angle $ACB$, and $I_1C$ bisects the angle $BCM$.

$$\therefore \quad \angle ICI_1 = \angle ICB + \angle I_1CB$$

$$= \frac{1}{2} \angle ACB + \frac{1}{2} \angle MCB$$

$$= \frac{1}{2} [ \angle ACB + \angle MCB]$$

$$= \frac{1}{2} \cdot 180^\circ = \text{a right angle.}$$

Similarly, $\angle ICI_2$ is a right angle.

Hence $I_1CI_2$ is a straight line to which $IC$ is perpendicular.

So $I_2AI_3$ is a straight line to which $IA$ is perpen-
dicular, and \( I_3B I_1 \) is a straight line to which \( IB \) is perpendicular.

Also, since \( IA \) and \( I_1A \) both bisect the angle \( BAC \), the three points \( A, I, \) and \( I_1 \) are in a straight line. Similarly \( BII_2 \) and \( CII_3 \) are straight lines. Hence \( I_1I_2I_3 \) is a triangle, which is such that \( A, B, \) and \( C \) are the feet of the perpendiculars drawn from its vertices upon the opposite sides, and such that \( I \) is the intersection of these perpendiculars, \textit{i.e.} \( ABC \) is its pedal triangle and \( I \) is its orthocentre.

The triangle \( I_1I_2I_3 \) is often called the excentric triangle.

212. Centroid and Medians of any Triangle.

If \( ABC \) be any triangle, and \( D, E, \) and \( F \) respectively the middle points of \( BC, CA, \) and \( AB \), the lines \( AD, BE, \) and \( CF \) are called the medians of the triangle.

It is shewn in most editions of Euclid that the medians meet in a common point \( G \), such that

\[
AG = \frac{2}{3} AD, \quad BG = \frac{2}{3} BE, \quad CG = \frac{2}{3} CF.
\]

This point \( G \) is called the centroid of the triangle.

213. Length of the medians. We have, by Art. 164,

\[
AD^2 = AC^2 + CD^2 - 2AC \cdot CD \cos C
\]

\[
= b^2 + \frac{a^2}{4} - ab \cos C,
\]

and

\[
c^2 = b^2 + a^2 - 2ab \cos C.
\]
Hence \[ 2AD^2 - c^2 = b^2 - \frac{a^2}{2}, \]

so that \[ AD = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}. \]

Hence also \[ AD = \frac{1}{2} \sqrt{2c^2 + 2a^2 - b^2}. \]  
(Art. 164.)

So also \[ BE = \frac{1}{2} \sqrt{2c^2 + 2a^2 - b^2}, \text{ and } CF = \frac{1}{2} \sqrt{2a^2 + 2b^2 - c^2}. \]

214. Angles that the median \( AD \) makes with the sides.

If the \( \angle BAD = \beta \), and \( \angle CAD = \gamma \), we have

\[ \frac{\sin \gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2x}. \]

\[ \therefore \sin \gamma = \frac{a \sin C}{2x} = \frac{a \sin C}{\sqrt{2b^2 + 2c^2 - a^2}}. \]

Similarly, \[ \sin \beta = \frac{a \sin B}{\sqrt{2b^2 + 2c^2 - a^2}}. \]

Again, if the \( \angle ADC \) be \( \theta \), we have

\[ \frac{\sin \theta}{\sin C} = \frac{AC}{AD} = \frac{b}{x}. \]

\[ \therefore \sin \theta = \frac{b \sin C}{x} = \frac{2b \sin C}{\sqrt{2b^2 + 2c^2 - a^2}}. \]

The angles that \( AD \) makes with the sides are therefore found.

215. The centroid lies on the line joining the circumcentre to the orthocentre.
Let $O$ and $P$ be the circumcentre and orthocentre respectively. Draw $OD$ and $PK$ perpendicular to $BC$.

Let $AD$ and $OP$ meet in $G$.

The triangles $OGD$ and $PGA$ are clearly equiangular.

Also, by Art. 200,

$$OD = R \cos A$$

and, by Art. 209,

$$AP = 2R \cos A.$$ 

Hence, by Euc. vi. 4,

$$\frac{AG}{GD} = \frac{AP}{OD} = 2.$$ 

The point $G$ is therefore the centroid of the triangle.

Also, by the same proposition,

$$\frac{OG}{GP} = \frac{OD}{AP} = \frac{1}{2}.$$ 

The centroid therefore lies on the line joining the circumcentre to the orthocentre, and divides it in the ratio $1:2$.

It may be shewn by geometry that the centre of the nine-point circle (which passes through the feet of the perpendiculars, the middle points of the sides, and the middle points of the lines joining the angular points to the orthocentre) lies on $OP$ and bisects it.

The circumcentre, the centroid, the centre of the nine-point circle, and the orthocentre therefore all lie on a straight line.

216. **Distance between the circumcentre and the orthocentre.**
If $OF$ be perpendicular to $AB$, we have
\[ \angle OAF = 90^\circ - \angle OAB = 90^\circ - C. \]
Also \[ \angle PAL = 90^\circ - C. \]
\[ \therefore \angle OAP = A - \angle OAF - \angle PAL \]
\[ = A - 2(90^\circ - C) = A + 2C - 180^\circ \]
\[ = A + 2C - (A + B + C) = C - B. \]
Also $OA = R$, and, by Art. 209,
\[ PA = 2R \cos A. \]
\[ \therefore OP^2 = OA^2 + PA^2 - 2OA \cdot PA \cos OAP \]
\[ = R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B) \]
\[ = R^2 - 4R^2 \cos A [\cos (B+C) + \cos (C - B)] \]
\[ = R^2 - 8R^2 \cos A \cos B \cos C. \]
\[ \therefore OP = R \sqrt{1 - 8 \cos A \cos B \cos C}. \]

*217. To find the distance between the circumcentre and the incentre.

Let $O$ be the circumcentre, and let $OF$ be perpendicular to $AB$.

Let $I$ be the incentre, and $IE$ be perpendicular to $AC$.

Then, as in the last article,
\[ \angle OAF = 90^\circ - C. \]
\[ \therefore \angle OAI = \angle IAF - \angle OAF \]
\[ = \frac{A}{2} - (90^\circ - C) = \frac{A}{2} + C - \frac{A + B + C}{2} = \frac{C - B}{2}. \]
Also \( A I = \frac{IE}{\sin \frac{A}{2}} = \frac{r}{\sin \frac{A}{2}} = 4R \sin \frac{B}{2} \sin \frac{C}{2} \) (Art 204. Cor.).

\[ \therefore OI^2 = OA^2 + AI^2 - 2OA \cdot AI \cos OAI \]

\[ = R^2 + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 8R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{C - B}{2} . \]

\[ \therefore \frac{OI^2}{R^2} = 1 + 16 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \]

\[ - 8 \sin \frac{B}{2} \sin \frac{C}{2} \left[ \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right] \]

\[ = 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \left( \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right) \]

\[ = 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{B + C}{2} \]

\[ = 1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{A}{2} . \quad \text{(Art. 69)} \ldots \ldots \ldots (1). \]

\[ \therefore OI = R \sqrt{1 - 8 \sin \frac{B}{2} \sin \frac{C}{2} \sin \frac{A}{2}} . \]

Also (1) may be written

\[ OI^2 = R^2 - 2R \times 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \]

\[ = R^2 - 2Rr . \quad \text{(Art. 204. Cor.)} \]

In a similar manner it may be shewn that, if \( I_1 \) be the centre of the escribed circle opposite the angle \( A \), we shall have

\[ OI_1 = R \sqrt{1 + 8 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} ,} \]

and hence \[ OI_1^2 = R^2 + 2Rr_1 . \quad \text{(Art. 207. Cor.)} \]
**Aliter.** Let $OI$ be produced to meet the circumcircle of the triangle in $S$ and $T$, and let $AI_1$ meet it in $II$. 
By Euc. iii. 35, we have 
\[ SI . IT = AI . IH \] 
\[ \text{But } SI . IT = (R + OI) (R - OI) = R^2 - OI^2. \]
\[ \text{Also } \angle HIC = \angle ICA + \angle IAC = \angle ICB + \angle HAB \]
\[ = \angle ICB + \angle HIC \]
\[ = \angle HIC. \]
\[ \therefore HI = HC = 2R \sin \frac{A}{2}. \quad \text{(Art. 200.)} \]
\[ \text{Also } AI = \frac{IE}{\sin \frac{A}{2}} = \frac{r}{\sin \frac{A}{2}}. \]
Substituting in (2), we have 
\[ R^2 - OI^2 = 2Rr, \]
i.e. 
\[ OI^2 = R^2 - 2Rr. \]
Similarly, we can shew that $I_1I = I_1C$, and hence that 
\[ I_1O^2 - R^2 = I_1H \cdot I_1A = 2Rr_1, \]
i.e. 
\[ I_1O^2 = R^2 + 2Rr_1. \]

**218. Bisectors of the angles.**

If $AD$ bisect the angle $A$ and divide the base into portions $x$ and $y$, we have, by Euc. vi. 3, 
\[ \frac{x}{c} = \frac{AB}{AC} = \frac{c}{b}. \]
\[ \therefore \frac{x}{c} = \frac{y}{b} = \frac{x + y}{b + c} = \frac{a}{b + c} \quad \ldots \quad (1), \]
giving $x$ and $y$.

Also, if $\delta$ be the length of $AD$ and $\theta$ the angle it makes with $BC$, we have 
\[ \triangle ABD + \triangle ACD = \triangle ABC. \]
\[ \therefore \frac{1}{2} c \delta \sin \frac{A}{2} + \frac{1}{2} b \delta \sin \frac{A}{2} = \frac{1}{2} bc \sin A, \]
i.e.  \[ \delta = \frac{bc}{b + c} \frac{\sin A}{\sin \frac{A}{2}} = \frac{2bc}{b + c} \cos \frac{A}{2} \]  \[ \cdots \cdots \cdots (2). \]

Also  \[ \theta = \angle DAB + B = \frac{A}{2} + B \]  \[ \cdots \cdots \cdots (3). \]

We thus have the length of the bisector and its inclination to \( BC \).

**EXAMPLES. XXXVII.**

If \( I, I_1, I_2, \) and \( I_3 \) be respectively the centres of the incircle and the three escribed circles of a triangle \( ABC \), prove that

1. \( AI = r \csc \frac{A}{2} \).
2. \( IA \cdot IB \cdot IC = abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \).
3. \( AI_1 = r_1 \csc \frac{A}{2} \).
4. \( II_1 = a \sec \frac{A}{2} \).
5. \( I_2 I_3 = a \csc \frac{A}{2} \).
6. \( II_1 \cdot II_2 \cdot II_3 = 16R^2r \).
7. \( I_2 I_3^2 = 4R (r_2 + r_3) \).
8. \( \angle I_3 I_1 I_2 = \frac{B + C}{2} \).
9. \( I_1^2 + I_2^2 + I_3^2 = II_1^2 + I_2^2 + I_3^2 + I_1 I_2^2 \).
10. Area of \( \triangle I_1 I_2 I_3 = 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{abc}{4r} \).
11. \( \frac{II_1}{\sin A} = \frac{II_2}{\sin B} = \frac{II_3}{\sin C} \).

If \( I, O, \) and \( P \) be respectively the incentre, circumcentre, and orthocentre, and \( G \) the centroid of the triangle \( ABC \), prove that

12. \( IO^2 = R^2 (3 - 2 \cos A - 2 \cos B - 2 \cos C) \).
13. \( IP^2 = 2R^2 - 4R^2 \cos A \cos B \cos C \).
14. \( OG^2 = R^2 - \frac{1}{9} (a^2 + b^2 + c^2) \).
15. Area of \( \triangle IOP = 2R^2 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2} \).
16. Area of $\Delta IPG = \frac{1}{3} R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$.

17. Prove that the distance of the centre of the nine-point circle from the angle $A$ is $\frac{R}{2} \sqrt{1 + 8 \cos A \sin B \sin C}$.

18. $DEF$ is the pedal triangle of $ABC$; prove that
   
   (1) its area is $2S \cos A \cos B \cos C$,
   
   (2) the radius of its circumcircle is $\frac{R}{2}$,
   
   and (3) the radius of its incircle is $2R \cos A \cos B \cos C$.

19. $O_1 O_2 O_3$ is the triangle formed by the centres of the escribed circles of the triangle $ABC$; prove that
   
   (1) its sides are $4R \cos \frac{A}{2}$, $4R \cos \frac{B}{2}$, and $4R \cos \frac{C}{2}$,
   
   (2) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$, and $\frac{\pi}{2} - \frac{C}{2}$,
   
   and (3) its area is $2R \sin S$.

20. $DEF$ is the triangle formed by joining the points of contact of the incircle with the sides of the triangle $ABC$; prove that
   
   (1) its sides are $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$, and $2r \cos \frac{C}{2}$,
   
   (2) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$, and $\frac{\pi}{2} - \frac{C}{2}$,
   
   and (3) its area is $\frac{2S^3}{abcs}$, i.e. $\frac{1}{2} R S$.

21. $D$, $E$, and $F$ are the middle points of the sides of the triangle $ABC$; prove that the centroid of the triangle $DEF$ is the same as that of $ABC$, and that its orthocentre is the circumcentre of $ABC$.

   In any triangle $ABC$, prove that

22. The perpendicular from $A$ divides $BC$ into portions which are proportional to the cotangents of the adjacent angles, and that it divides the angle $A$ into portions whose cosines are inversely proportional to the adjacent sides.

23. The median through $A$ divides it into angles whose cotangents are $2 \cot A + \cot C$ and $2 \cot A + \cot B$, and makes with the base an angle whose cotangent is $\frac{1}{2} (\cot C - \cot B)$. 
24. The distance between the middle point of $BC$ and the foot of the perpendicular from $A$ is $\frac{b^2 - c^2}{2a}$.

25. $O$ is the orthocentre of a triangle $ABC$; prove that the radii of the circles circumscribing the triangles $BOC$, $COA$, $AOB$, and $ABC$ are all equal.

26. $AD$, $BE$, and $CF$ are the perpendiculars from the angular points of a triangle $ABC$ upon the opposite sides; prove that the diameters of the circumscribed circles of the triangles $AEF$, $BDF$, and $CDE$ are respectively $a \cot A$, $b \cot B$, and $c \cot C$, and that the perimeters of the triangles $DEF$ and $ABC$ are in the ratio $t : R$.

27. Prove that the product of the distances of the incentre from the angular points of a triangle is $4Rr^2$.

28. The triangle $DEF$ circumscribes the three escribed circles of the triangle $ABC$; prove that

$$\frac{EF}{a \cos A} = \frac{FD}{b \cos B} = \frac{DE}{c \cos C}.$$

29. If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side $BC$ externally, prove that its radius is

$$\frac{\Delta}{a \tan^2 \frac{A}{2}}.$$

30. If $a$, $b$, and $c$ be the radii of three circles which touch one another externally, and $r_1$ and $r_2$ be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

31. If $\Delta_0$ be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is $\Delta$, and $\Delta_1$, $\Delta_2$, and $\Delta_3$ the corresponding areas for the escribed circles, prove that

$$\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta.$$

32. If the bisectors of the angles of a triangle $ABC$ meet the opposite sides in $A'$, $B'$, and $C'$, prove that the ratio of the areas of the triangles $A'B'C'$ and $ABC$ is

$$2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} : \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2}.$$
33. Through the angular points of a triangle are drawn straight lines which make the same angle $\theta$ with the opposite sides of the triangle; prove that the area of the triangle formed by them is to the area of the original triangle as $4 \cos^2 \theta : 1$.

34. Two circles, of radii $a$ and $b$, cut each other at an angle $\theta$. Prove that the length of the common chord is
\[
\frac{2ab \sin \theta}{\sqrt{a^2 + b^2 + 2ab \cos \theta}}.
\]

35. Three equal circles touch one another; find the radius of the circle which touches all three.

36. Three circles, whose radii are $a$, $b$, and $c$, touch one another externally and the tangents at their points of contact meet in a point; prove that the distance of this point from either of their points of contact is
\[
\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.
\]

37. In the sides $BC$, $CA$, $AB$ are taken three points $A'$, $B'$, $C'$ such that
\[
BA' : A'C = CB' : B'A = AC' : C'B = m : n;
\]
prove that if $AA'$, $BB'$, and $CC'$ be joined they will form by their intersections a triangle whose area is to that of the triangle $ABC$ as
\[
(m-n)^2 : m^2 + mn + n^2.
\]

38. The circle inscribed in the triangle $ABC$ touches the sides $BC$, $CA$, and $AB$ in the points $A_1$, $B_1$, and $C_1$ respectively; similarly the circle inscribed in the triangle $A_1B_1C_1$ touches the sides in $A_2$, $B_2$, $C_2$ respectively, and so on; if $A_nB_nC_n$ be the $n$th triangle so formed, prove that its angles are
\[
\frac{\pi}{3} + (-2)^{-n}(A - \frac{\pi}{3}), \quad \frac{\pi}{3} + (-2)^{-n}(B - \frac{\pi}{3}),
\]
and
\[
\frac{\pi}{3} + (-2)^{-n}(C - \frac{\pi}{3}).
\]
Hence prove that the triangle so formed is ultimately equilateral.

39. $A_1B_1C_1$ is the triangle formed by joining the feet of the perpendiculars drawn from $ABC$ upon the opposite sides; in like manner $A_2B_2C_2$ is the triangle obtained by joining the feet of the perpendiculars from $A_1$, $B_1$, and $C_1$ on the opposite sides, and so on. Find the values of the angles $A_n$, $B_n$, and $C_n$ in the $n$th of these triangles.
CHAPTER XVI.

ON QUADRILATERALS AND REGULAR POLYGONS.

219. To find the area of a quadrilateral which is inscribable in a circle.

Let $ABCD$ be the quadrilateral, the sides being $a$, $b$, $c$, and $d$ as marked in the figure.

The area of the quadrilateral

$= \text{area of } \triangle ABC + \text{area of } \triangle ACD$

$= \frac{1}{2}absinB + \frac{1}{2}cd \sin D$ (Art. 198.)

$= \frac{1}{2} (ab + cd) \sin B$,

since, by Euc. III. 22,

$\angle B = 180^\circ - \angle D$,

and therefore

$\sin B = \sin D$.

We have to express $\sin B$ in terms of the sides.

We have

$a^2 + b^2 - 2ab \cos B = AC^2 = c^2 + d^2 - 2cd \cos D$.

But $\cos D = \cos (180^\circ - B) = - \cos B$. 
Hence
\[ a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B, \]
so that \[ \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}. \]
Hence
\[ \sin^2 B = 1 - \cos^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}. \]
\[ = \frac{\left\{2(ab + cd)\right\}^2 - (a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2} \]
\[ = \frac{\left\{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\right\} \left\{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\right\}}{4(ab + cd)^2} \]
\[ = \frac{\left\{(a^2 + 2ab + b^2) - (c^2 - 2cd + d^2)\right\} \left\{(a^2 + 2cd + d^2) - (a^2 + b^2 - 2ab)\right\}}{4(ab + cd)^2} \]
\[ = \frac{\left\{(a + b)^2 - (c - d)^2\right\} \left\{(c + d)^2 - (a - b)^2\right\}}{4(ab + cd)^2} \]
\[ = \frac{\left\{(a + b + c - d)(a + b - c + d)\right\} \left\{(c + d + a - b)\right\}(c + d - a + b)}{4(ab + cd)^2} \]
Let \( a + b + c + d = 2s, \)
so that
\( a + b + c - d = (a + b + c + d) - 2d = 2(s - d), \)
\( a + b - c + d = 2(s - c), \)
\( a - b + c + d = 2(s - b), \)
and \( -a + b + c + d = 2(s - a). \)
Hence
\[ \sin^2 B = \frac{2(s - d) \times 2(s - c) \times 2(s - b) \times 2(s - a)}{4(ab + cd)^2}, \]
so that
\[ (ab + cd) \sin B = 2\sqrt{(s - a)(s - b)(s - c)(s - d)}. \]
Hence the area of the quadrilateral
\[ = \frac{1}{2}(ab + cd) \sin B = \sqrt{(s - a)(s - b)(s - c)(s - d)}. \]
220. Since \( \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \),

we have \( AC^2 = a^2 + b^2 - 2ab \cos B \)

\[ = a^2 + b^2 - ab \left( \frac{a^2 + b^2 - c^2 - d^2}{ab + cd} \right) \]

\[ = \frac{(a^2 + b^2)cd + ab(c^2 + d^2)}{ab + cd} \]

\[ = \frac{(ac + bd)(ad + bc)}{ab + cd}. \]

Similarly it could be proved that

\( BD^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}. \)

We thus have the lengths of the diagonals of the quadrilateral.

It follows by multiplication that

\( AC^2 \cdot BD^2 = (ac + bd)^2, \)

i.e. \( AC \cdot BD = AB \cdot CD + BC \cdot AD. \)

This is Euc. vi. Prop. D.

Again, the radius of the circle inscribing the quadrilateral \( = \frac{AC}{\sin B} \)

\[ = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}} \div 4 \sqrt{\frac{(s - a)(s - b)(s - c)(s - d)}{(ab + cd)^2}} \]

\[ = \frac{1}{4} \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}. \]

221. If we have any quadrilateral, not necessarily inscribable in a circle, we can express its area in terms of its sides and the sum of any two opposite angles.
For let the sum of the two angles $B$ and $D$ be denoted by $2\alpha$, and denote the area of the quadrilateral by $\Delta$.

Then

$$\Delta = \text{area of } ABC + \text{area of } ACD = \frac{1}{2}ab \sin B + \frac{1}{2}cd \sin D,$$

so that

$$4\Delta = 2ab \sin B + 2cd \sin D \ldots (1).$$

Also

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D,$$

so that

$$a^2 + b^2 - c^2 - d^2 = 2ab \cos B - 2cd \cos D \ldots (2).$$

Squaring (1) and (2) and adding, we have

$$16\Delta^2 + (a^2 + b^2 - c^2 - d^2)^2 = 4a^2b^2 + 4c^2d^2 - 8abcd \cos (B + D)$$

$$= 4a^2b^2 + 4c^2d^2 - 8abcd \cos 2\alpha$$

$$= 4a^2b^2 + 4c^2d^2 - 8abcd (2 \cos^2 \alpha - 1)$$

$$= 4(ab + cd)^2 - 16abcd \cos^2 \alpha,$$

so that

$$16\Delta^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd \cos^2 \alpha$$

$$\ldots \ldots \ldots \ldots (3).$$

But, as in Art. 219, we have

$$4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$= 2(s - a) \cdot 2(s - b) \cdot 2(s - c) \cdot 2(s - d)$$

$$= 16(s - a)(s - b)(s - c)(s - d).$$

Hence (3) becomes

$$\Delta^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \alpha,$$

giving the required area.

**Cor. 1.** If $d$ be zero, the quadrilateral becomes a triangle, and the formula above becomes that of Art. 198.
Cor. 2. If the sides of the quadrilateral be given in length, we know \(a, b, c, d\) and therefore \(s\). The area \(\Delta\) is hence greatest when \(abcd \cos^2 \alpha\) is least, that is when \(\cos^2 \alpha\) is zero, and then \(\alpha = 90^\circ\). In this case the sum of two opposite angles of the quadrilateral is \(180^\circ\) and the figure inscribable in a circle. (Euc. III. 22.)

The quadrilateral, whose sides are given, has therefore the greatest area when it can be inscribed in a circle.

222. Ex. Find the area of a quadrilateral which can have a circle inscribed in it.

If the quadrilateral \(ABCD\) can have a circle inscribed in it so as to touch the sides \(AB, BC, CD,\) and \(DA\) in the points \(P, Q, R,\) and \(S,\) we should have

\[
AP = AS, \quad BP = BQ, \quad CQ = CR, \quad \text{and} \quad DR = DS.
\]

\[
\therefore \quad AP + BP + CR + DR = AS + BQ + CQ + DS,
\]

\[
i.e. \quad AB + CD = BC + DA,
\]

\[
i.e. \quad a + c = b + d.
\]

Hence

\[
s = \frac{a + b + c + d}{2} = a + c = b + d.
\]

\[
\therefore \quad s = a = c, \quad s - b = d, \quad s - c = a, \quad \text{and} \quad s - d = b.
\]

The formula of the last article therefore gives in this case

\[
\Delta^2 = abcd - abcd \cos^2 \alpha = abcd \sin^2 \alpha,
\]

\[
i.e. \quad \text{the area required} = \sqrt{abcd} \sin \alpha.
\]

If in addition the quadrilateral be also inscribable in a circle, we have

\[
2a = 180^\circ, \quad \text{so that} \quad \sin a = \sin 90^\circ = 1.
\]

Hence the area of a quadrilateral which can be both inscribed in a circle and circumscribed about another circle is \(\sqrt{abcd}d\).

EXAMPLES. XXXVIII.

1. Find the area of a quadrilateral, which can be inscribed in a circle, whose sides are

(1) 3, 5, 7, and 9 feet;

and

(2) 7, 10, 5, and 2 feet.

2. The sides of a quadrilateral are respectively 3, 4, 5, and 6 feet, and the sum of a pair of opposite angles is \(120^\circ\); prove that the area of the quadrilateral is \(3\sqrt{30}\) square feet.
3. The sides of a quadrilateral which can be inscribed in a circle are 3, 3, 4, and 4 feet; find the radii of the incircle and circumcircle.

4. Prove that the area of any quadrilateral is one-half the product of the two diagonals and the sine of the angle between them.

5. If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is $\sqrt{abcd}$, and that the radius of the latter circle is

$$\frac{2\sqrt{abcd}}{a+b+c+d}$$

6. A quadrilateral $ABCD$ is described about a circle; prove that

$$AB \sin \frac{A}{2} \sin \frac{B}{2} = CD \sin \frac{C}{2} \sin \frac{D}{2}.$$ 

7. $a, b, c,$ and $d$ are the sides of a quadrilateral taken in order, and $a$ is the angle between the diagonals opposite to $b$ or $d$; prove that the area of the quadrilateral is

$$\frac{1}{4} (a^2 - b^2 + c^2 - d^2) \tan a.$$ 

8. If $a, b, c,$ and $d$ be the sides and $x$ and $y$ the diagonals of a quadrilateral, prove that its area is

$$\frac{1}{4} \left[ 4xy^2 - (b^2 + d^2 - a^2 - c^2)^2 \right]^{\frac{1}{4}}.$$ 

9. If a quadrilateral can be inscribed in a circle, prove that the angle between its diagonals is

$$\sin^{-1} \left[ 2\sqrt{(s-a)(s-b)(s-c)(s-d)} : (ac+bd) \right].$$

If the same quadrilateral can also be circumscribed about a circle, prove that this angle is then

$$\cos^{-1} \frac{ac-bd}{ac+bd}.$$ 

10. The sides of a quadrilateral are divided in order in the ratio $m:n$, and a new quadrilateral is formed by joining the points of division; prove that its area is to the area of the original figure as $m^2 + n^2$ to $(m+n)^2$.

11. If $ABCD$ be a quadrilateral inscribed in a circle, prove that

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}},$$

and that the product of the segments into which one diagonal is divided by the other diagonal is

$$\frac{abcd}{(ab+cd)(ad+bc)}. $$
12. If \( a, b, c, \) and \( d \) be the sides of a quadrilateral, taken in order, prove that
\[
d^2 = a^2 + b^2 + c^2 - 2ab \cos \alpha - 2bc \cos \beta - 2ca \cos \gamma,
\]
where \( \alpha, \beta, \) and \( \gamma \) denote the angles between the sides \( a \) and \( b, b \) and \( c, \)
and \( c \) and \( a \) respectively.

223. **Regular Polygons.** A regular polygon is a polygon which has all its sides equal and all its angles equal.

If the polygon have \( n \) angles we have, by *Eucl. i. 32, Cor., \( n \) times its angle + 4 right angles = twice as many right angles as the figure has sides = \( 2n \) right angles.

Hence each angle = \( \frac{2n - 4}{n} \) right angles = \( \frac{2n - 4}{n} \times \frac{\pi}{2} \) radians.

224. **Radii of the inscribed and circumscribing circles of a regular polygon.**

Let \( AB, BC, \) and \( CD \) be three successive sides of the polygon, and let \( n \) be the number of its sides.

Bisect the angles \( ABC \) and \( BCD \) by the lines \( BO \) and \( CO \) which meet in \( O, \) and draw \( OL \) perpendicular to \( BC. \)

It is easily seen that \( O \) is the centre of both the incircle and the circumcircle of the polygon, and that \( BL \) equals \( LC. \)

Hence we have \( OB = OC = R, \) the radius of the circumcircle, and \( OL = r, \) the radius of the incircle.

L. T.
The angle $BOC$ is $\frac{1}{n}$th of the sum of all the angles subtended at $O$ by the sides, i.e.

$$\angle BOC = \frac{4 \text{ right angles}}{n} = \frac{2\pi}{n} \text{ radians}.$$ 

Hence

$$\angle BOL = \frac{1}{2} \angle BOC = \frac{\pi}{n}.$$ 

If $a$ be a side of the polygon, we have

$$a = BC = 2BL = 2R \sin BOL = 2R \sin \frac{\pi}{n}.$$ 

Therefore

$$R = \frac{a}{2 \sin \frac{\pi}{n}} = \frac{a}{2} \csc \frac{\pi}{n} \text{ ..........(1)}.$$ 

Again, $a = 2BL = 2OL \tan BOL = 2r \tan \frac{\pi}{n}.$

Therefore

$$r = \frac{a}{2 \tan \frac{\pi}{n}} = \frac{a}{2} \cot \frac{\pi}{n} \text{ ..........(2)}.$$ 

225. **Area of a Regular Polygon.**

The area of the polygon is $n$ times the area of the triangle $BOC$.

Hence the area of the polygon

$$= n \times \frac{1}{2} OL \cdot BC = n \cdot OL \cdot BL = n \cdot BL \cot LOB \cdot BL = n \cdot \frac{a^2}{4} \cot \frac{\pi}{n} \text{ ......(1)},$$

an expression for the area in terms of the side.

Also the area

$$= n \cdot OL \cdot BL = n \cdot OL \cdot OL \tan BOL = nr^2 \tan \frac{\pi}{n} \text{ ..........(2)}.$$
Again, the area
\[ = n \cdot OL \cdot BL = n \cdot OB \cos LOB \cdot OB \sin LOB \]
\[ = nR^2 \cos \frac{\pi}{n} \sin \frac{\pi}{n} = \frac{n}{2} R^2 \sin \frac{2\pi}{n} \]
\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots (3). \]

The formulae (2) and (3) give the area in terms of the radius of the inscribed and circumscribed circles.

226. Ex. The length of each side of a regular dodecagon is 20 feet; find (1) the radius of its inscribed circle, (2) the radius of its circumscribing circle, and (3) its area.

The angle subtended by a side at the centre of the polygon
\[ = \frac{360^\circ}{12} = 30^\circ. \]

Hence we have
\[ 10 = r \tan 15^\circ = R \sin 15^\circ. \]
\[ \therefore \ r = 10 \cot 15^\circ \]
\[ = \frac{10}{2 - \sqrt{3}} \quad \text{(Art. 101)} \]
\[ = 10 \left(2 + \sqrt{3}\right) \approx 37.32\ldots \text{ feet.} \]

Also
\[ R = \frac{10}{\sin 15^\circ} = 10 \times \frac{2 \sqrt{2}}{\sqrt{3} - 1} \quad \text{(Art. 106)} \]
\[ = 10 \cdot \frac{2 \sqrt{2}}{\sqrt{3} + 1} = 10 \left(\sqrt{6} + \sqrt{2}\right) \]
\[ = 10 \left(2.4142\ldots + 1.4142\ldots\right) = 38.637 \ldots \text{ feet.} \]

Again, the area
\[ = 12 \times r \times 10 \text{ square feet} \]
\[ = 1200 \left(2 + \sqrt{3}\right) = 447.16\ldots \text{ square feet.} \]

EXAMPLES. XXXIX.

1. Find, correct to 0.01 of an inch, the length of the perimeter of a regular decagon which surrounds a circle of radius one foot. \[ \tan 18^\circ = 0.32492. \]

2. Find to 3 places of decimals the length of the side of a regular polygon of 12 sides which is circumscribed to a circle of unit radius.

3. Find the area of (1) a pentagon, (2) a hexagon, (3) an octagon, (4) a decagon and (5) a dodecagon, each being a regular figure of side 1 foot. \[ \cot 18^\circ = 3.07763\ldots; \quad \cot 36^\circ = 1.37638\ldots. \]

4. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each be 24 feet.
5. A square, whose side is 2 feet, has its corners cut away so as to form a regular octagon; find its area.

6. Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle, and shew that the areas of the inscribed hexagon and octagon are as $\sqrt{27}$ to $\sqrt{32}$.

7. Prove that the radius of the circle described about a regular pentagon is nearly $\frac{2}{3}$ths of the side of the pentagon.

8. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as $2:3$.

9. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2:\sqrt{5}$.

10. Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of $n$ sides, is

$$\frac{a}{2} \cot \frac{\pi}{2n},$$

where $a$ is a side of the polygon.

11. Of two regular polygons of $n$ sides, one circumscribes and the other is inscribed in a given circle. Prove that the perimeters of the circumscribing polygon, the circle, and the inscribed polygon are in the ratio

$$\sec \frac{\pi}{n} : \frac{\pi}{n} \cosec \frac{\pi}{n} : 1,$$

and that the areas of the polygons are in the ratio $\cos^2 \frac{\pi}{n} : 1$.

12. Given that the area of a polygon of $n$ sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3:2$, find $n$.

13. Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of $n$ sides.

14. The area of a regular polygon of $n$ sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 is to 4. Find the value of $n$.

15. The interior angles of a polygon are in A.P.; the least angle is $120^\circ$ and the common difference is $5^\circ$; find the number of sides.
16. There are two regular polygons the number of sides in one being
double the number in the other, and an angle of one polygon is to an angle
of the other as 9 to 8; find the number of sides of each polygon.

17. Show that there are eleven pairs of regular polygons such that
the number of degrees in the angle of one is to the number in the angle of
the other as 10:9. Find the number of sides in each.

18. The side of a base of a square pyramid is \(a\) feet and its vertex is
at a height of \(h\) feet above the centre of the base; if \(\theta\) and \(\phi\) be respec-
tively the inclinations of any face to the base, and of any two faces to one
another, prove that

\[
\tan \theta = \frac{2h}{a} \quad \text{and} \quad \tan \frac{\phi}{2} = \sqrt{1 + \frac{a^2}{2h^2}}.
\]

19. A pyramid stands on a regular hexagon as base. The perpendi-
cular from the vertex of the pyramid on the base passes through the
centre of the hexagon, and its length is equal to that of a side of the base.
Find the tangent of the angle between the base and any face of the
pyramid, and also of half the angle between any two side faces.

20. A regular pyramid has for its base a polygon of \(n\) sides, each of
length \(a\), and the length of each slant side is \(l\); prove that the cosine of
the angle between two adjacent lateral faces is

\[
\frac{4l^2 \cos \frac{2\pi}{n} + a^2}{4l^2 - a^2}.
\]
CHAPTER XVII.

TRIGONOMETRICAL RATIOS OF SMALL ANGLES. AREA OF A CIRCLE. DIP OF THE HORIZON.

227. If $\theta$ be the number of radians in any angle, which is less than a right angle, then $\sin \theta$, $\theta$, and $\tan \theta$ are in ascending order of magnitude.

Let $TOP$ be any angle which is less than a right angle.

With centre $O$ and any radius $OP$ describe an arc $PAP'$ meeting $OT$ in $A$.

Draw $PN$ perpendicular to $OA$, and produce it to meet the arc of the circle in $P'$.

Draw the tangent $PT$ at $P$ to meet $OA$ in $T$, and join $TP'$.

The triangles $PON$ and $P'ON$ are equal in all respects, so that $PN = NP'$ and $\text{arc } PA = \text{arc } AP'$.

Also the triangles $TOP$ and $TOP'$ are equal in all respects, so that $TP = TP'$. 
The straight line $PP'$ is less than the arc $PAP'$, so that $NP$ is $<$ arc $PA$.

We shall assume that the arc $PAP'$ is less than the sum of $PT$ and $TP'$, so that arc $PA < PT$.

Hence $NP$, the arc $AP$, and $PT$ are in ascending order of magnitude.

Therefore $\frac{NP}{OP}$, $\frac{AP}{OP}$, and $\frac{PT}{OP}$ are in ascending order of magnitude.

But $\frac{NP}{OP} = \sin AOP = \sin \theta$,

$\frac{AP}{OP} = \text{number of radians in } \angle AOP = \theta \text{ (Art. 21)}$,

and $\frac{PT}{OP} = \tan POT = \tan AOP = \tan \theta$.

Hence $\sin \theta$, $\theta$, and $\tan \theta$ are in ascending order of magnitude, provided that

$\theta < \frac{\pi}{2}$.

228. Since $\sin \theta < \theta < \tan \theta$, we have, by dividing each by the positive quantity $\sin \theta$,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$ 

Hence $\frac{\theta}{\sin \theta}$ always lies between 1 and $\frac{1}{\cos \theta}$.

This holds however small $\theta$ may be.

Now, when $\theta$ is very small, $\cos \theta$ is very nearly unity, and the smaller $\theta$ becomes, the more nearly does $\cos \theta$ become unity, and hence the more nearly does $\frac{1}{\cos \theta}$ become unity.
Hence, when \( \theta \) is very small, the quantity \( \frac{\theta}{\sin \theta} \) lies between 1 and a quantity which differs from unity by an indefinitely small quantity.

In other words, when \( \theta \) is made indefinitely small the quantity \( \frac{\theta}{\sin \theta} \), and therefore \( \frac{\sin \theta}{\theta} \), is ultimately equal to unity, i.e. the smaller an angle becomes the more nearly is its sine equal to the number of radians in it.

This is often shortly expressed thus;

\[
\sin \theta = \theta, \text{ when } \theta \text{ is very small.}
\]

So also \( \tan \theta = \theta \), when \( \theta \) is very small.

**Cor.** Putting \( \theta = \frac{\alpha}{n} \), it follows that, when \( \theta \) is indefinitely small, \( n \) is indefinitely great.

\[
\sin \frac{\alpha}{n}
\]

Hence \( \frac{\alpha}{n} \) is unity, when \( n \) is indefinitely great.

\[
\frac{\alpha}{n}
\]

So \( n \sin \frac{\alpha}{n} = \alpha \), when \( n \) is indefinitely great.

Similarly, \( n \tan \frac{\alpha}{n} = \alpha \), when \( n \) is indefinitely great.

229. In the preceding article it must be particularly noticed that \( \theta \) is the number of radians in the angle considered.

The value of \( \sin \alpha^\circ \), when \( \alpha \) is small, may be found. For, since \( \pi^\circ = 180^\circ \), we have

\[
\alpha^\circ = \left(\pi \frac{\alpha}{180}\right)^\circ.
\]

\[
\therefore \sin \alpha^\circ = \sin \left(\pi \frac{\alpha}{180}\right)^\circ = \frac{\pi \alpha}{180},
\]

by the result of the last article.
230. From the tables it will be seen that the sine of an angle and its circular measure agree to 7 places of decimals so long as the angle is not greater than 18°. They agree to the 5th place of decimals so long as the angle is less than about 2°.

231. If θ be the number of radians in an angle, which is less than a right angle, then \( \sin \theta > \theta - \frac{\theta^3}{4} \) and \( \cos \theta > 1 - \frac{\theta^2}{2} \).

By Art. 227, we have

\[
\tan \frac{\theta}{2} > \frac{\theta}{2}.
\]

\[=\sin \frac{\theta}{2} > \frac{\theta}{2} \cos \frac{\theta}{2}.
\]

Hence, since \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \),

we have \( \sin \theta > \theta \cos \frac{\theta}{2} \), i.e. \( > \theta \left( 1 - \sin^2 \frac{\theta}{2} \right) \).

But since, by Art. 227,

\[
\sin \frac{\theta}{2} < \frac{\theta}{2},
\]

therefore \( 1 - \sin^2 \frac{\theta}{2} > 1 - \left( \frac{\theta}{2} \right)^2 \), i.e. \( > 1 - \frac{\theta^2}{4} \).

\[=\sin \theta > \theta \left( 1 - \frac{\theta^2}{4} \right), \ i.e. \ > \theta - \frac{\theta^3}{4}.
\]

Again,

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2},
\]

therefore, since \( \sin^2 \frac{\theta}{2} < \left( \frac{\theta}{2} \right)^2 \),

we have \( 1 - 2 \sin^2 \frac{\theta}{2} > 1 - 2 \left( \frac{\theta}{2} \right)^2 \), i.e. \( > 1 - \frac{\theta^2}{2} \).

It will be proved in Part II. that

\[\sin \theta > \theta - \frac{\theta^3}{6}, \text{ and } \cos \theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.
\]

232. Ex. 1. Find the values of \( \sin 10° \) and \( \cos 10° \).

Since \( 10° = \frac{10°}{6} = \frac{\pi°}{180 \times 6°} \),
we have

$$\sin 10' = \sin \left( \frac{\pi}{180 \times 6} \right) = \frac{\pi}{180 \times 6}$$

$$= \frac{3.14159265...}{180 \times 6} = 0.0029089 \text{ nearly.}$$

Also

$$\cos 10' = \sqrt{1 - \sin^2 10'}$$

$$= [1 - 0.000008168...]^\frac{1}{2}$$

$$= 1 - \frac{1}{2} [0.000008168...]$$

approximately by the Binomial Theorem,

$$= 1 - 0.000001234...$$

$$= 0.9999958...$$

**Ex. 2.** Solve approximately the equation

$$\sin \theta = 0.52.$$ 

Since $\sin \theta$ is very nearly equal to $\frac{1}{2}$, $\theta$ must be nearly equal to $\frac{\pi}{6}$.

Let then $\theta = \frac{\pi}{6} + x$, where $x$ is small.

$$\therefore \quad 0.52 = \sin \left( \frac{\pi}{6} + x \right) = \sin \frac{\pi}{6} \cos x + \cos \frac{\pi}{6} \sin x$$

$$= \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x.$$ 

Since $x$ is very small, we have

$$\cos x = 1 \text{ and } \sin x = x \text{ nearly.}$$

$$\therefore \quad 0.52 = \frac{1}{2} + \frac{\sqrt{3}}{2} x.$$ 

$$\therefore \quad x = 0.02 \times \frac{2}{\sqrt{3}} \text{ radians} = \frac{\sqrt{3} \times 2}{70} = 1.32^\circ \text{ nearly.}$$

Hence

$$\theta = 31^\circ 19' \text{ nearly.}$$

**EXAMPLES. XL.**

[\pi = 3.14159265; \quad \frac{1}{\pi} = 0.31831...]

Find, to 5 places of decimals, the value of

1. $\sin 7'$.  
2. $\sin 15''$.  
3. $\sin 1'$.  
4. $\cos 15'$.  
5. $\cosec 8''$.  
6. $\sec 5'$.  

Solve approximately the equations

7. \( \sin \theta = 0.01 \) 
8. \( \sin \theta = 0.43 \)

9. \( \cos \left( \frac{\pi}{3} + \theta \right) = 0.49 \) 
10. \( \cos \theta = 0.999 \)

11. Find approximately the distance at which a halfpenny, which is an inch in diameter, must be placed so as to just hide the moon, the angular diameter of the moon, that is the angle its diameter subtends at the observer's eye, being taken to be 30'.

12. A person walks in a straight line toward a very distant object, and observes that at three points \( A, B, \) and \( C \) the angles of elevation of the top of the object are \( a, 2a, \) and \( 3a \) respectively; prove that

\[ AB = 3BC \] nearly.

13. If \( \theta \) be the number of radians in an angle which is less than a right angle, prove that

\[ \cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} \]

14. Prove the theorem of Euler, viz. that

\[ \sin \theta = \theta \cdot \cos \frac{\theta}{2} \cdot \cos \frac{\theta}{2^2} \cdot \cos \frac{\theta}{2^3} \ldots \text{ad. inf.} \]

We have

\[ \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2^2 \sin \frac{\theta}{2^2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2} = \ldots \]

\[ = 2^n \sin \frac{\theta}{2^n} \times \cos \frac{\theta}{2} \cdot \cos \frac{\theta}{2^2} \cdot \cos \frac{\theta}{2^3} \ldots \cos \frac{\theta}{2^n}. \]

Make \( n \) indefinitely great so that, by Art. 228 Cor.,

\[ 2^n \sin \frac{\theta}{2^n} = \theta. \]

Hence

\[ \sin \theta = \theta \cdot \cos \frac{\theta}{2} \cdot \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \ldots \text{ad inf.} \]

15. Prove that

\[ \left( 1 - \tan^2 \frac{\theta}{2} \right) \left( 1 - \tan^2 \frac{\theta}{2^2} \right) \left( 1 - \tan^2 \frac{\theta}{2^3} \right) \ldots \text{ad inf.} = \theta \cdot \cot \theta. \]
233. **Area of a circle.**

By Art. 225, the area of a regular polygon of \( n \) sides, which is inscribed in a circle of radius \( R \), is

\[
\frac{n}{2} R^2 \sin \frac{2\pi}{n}.
\]

Let now the number of sides of this polygon be indefinitely increased, the polygon always remaining regular.

It is clear that the perimeter of the polygon must more and more approximate to the circumference of the circle.

Hence, when the number of sides of the polygon is infinitely great, the area of the circle must be the same as that of the polygon.

Now \( \frac{n}{2} R^2 \sin \frac{2\pi}{n} = \frac{n}{2} R^2 \cdot \frac{2\pi}{n} \cdot \sin \frac{2\pi}{n} = \pi R^2 \cdot \frac{2\pi}{n} \cdot \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2 \cdot \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2 \cdot \frac{\sin \theta}{\theta}, \) where \( \theta = \frac{2\pi}{n} \).

When \( n \) is made infinitely great, the value of \( \theta \) becomes infinitely small, and then, by Art. 228, \( \frac{\sin \theta}{\theta} \) is unity.

The area of the circle therefore = \( \pi R^2 = \pi \) times the square of its radius.

234. **Area of the sector of a circle.**

Let \( O \) be the centre of a circle, \( AB \) the bounding arc of the sector, and let \( \angle AOB = \alpha \) radians.

By Eucl. vi. 33, since sectors are to one another as the arcs on which they stand, we have

\[
\frac{\text{area of sector } AOB}{\text{area of whole circle}} = \frac{\text{arc } AB}{\text{circumference}} = \frac{R\alpha}{2\pi R} = \frac{\alpha}{2\pi}.
\]
\[ \text{area of sector } AO\theta = \frac{\alpha}{2\pi} \times \text{area of whole circle} \]
\[ = \frac{\alpha}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \cdot \alpha. \]

\**EXAMPLES. XLI.**

\[ \text{Assume that } \pi = 3.14159..., \quad \frac{1}{\pi} = 0.31831 \text{ and } \log \pi = 0.49715. \]

1. Find the area of a circle whose circumference is 71 feet.

2. The diameter of a circle is 10 feet; find the area of a sector whose arc is 22^\circ.

3. The area of a certain sector of a circle is 10 square feet; if the radius of the circle be 3 feet, find the angle of the sector.

4. The perimeter of a certain sector of a circle is 10 feet; if the radius of the circle be 3 feet, find the area of the sector.

5. A strip of paper, two miles long and \(0.003\) of an inch thick, is rolled up into a solid cylinder; find approximately the radius of the circular ends of the cylinder.

6. A strip of paper, one mile long, is rolled tightly up into a solid cylinder, the diameter of whose circular ends is 6 inches; find the thickness of the paper.

7. Given two concentric circles of radii \(r\) and \(2r\); two parallel tangents to the inner circle cut off an arc from the outer circle; find its length.

8. The circumference of a semicircle is divided into two arcs such that the chord of one is double that of the other. Prove that the sum of the areas of the two segments cut off by these chords is to the area of the semicircle as 27 is to 55.
\[ \pi = \frac{22}{7}. \]

9. If each of three circles, of radius \(a\), touch the other two, prove that the area included between them is nearly equal to \(\frac{4}{25}a^2\).
10. Six equal circles, each of radius \( a \), are placed so that each touches two others, their centres being all on the circumference of another circle; prove that the area which they enclose is
\[
2a^2(3\sqrt{3} - \pi).
\]

11. From the vertex \( A \) of a triangle \( \Delta \) a straight line \( AD \) is drawn making an angle \( \theta \) with the base and meeting it at \( D \). Prove that the area common to the circumscribing circles of the triangles \( ABD \) and \( ACD \) is
\[
\frac{1}{4}(b^2\gamma + c^2\beta - bc \sin A) \csc^2\theta,
\]
where \( \beta \) and \( \gamma \) are the number of radians in the angles \( B \) and \( C \) respectively.


Let \( O \) be a point at a distance \( h \) above the earth's surface. Draw tangents, such as \( OT \) and \( OT'' \), to the surface of the earth. The ends of all these tangents all clearly lie on a circle. This circle is called the Offing or Visible Horizon. The angle that each of these tangents \( OT \) makes with a horizontal plane \( POQ \) is called the Dip of the Horizon.

Let \( r \) be the radius of the earth, and let \( B \) be the other end of the diameter through \( A \).

We then have, by Eucl. III. 36,
\[
OT^2 = OA \cdot OB = h(2r + h),
\]
so that
\[
OT = \sqrt{h(2r + h)}.
\]

This gives an accurate value for \( OT \).

In all practical cases, however, \( h \) is very small compared with \( r \).

\([r = 4000 \text{ miles nearly, and } h \text{ is never greater, and generally is very considerably less, than } 5 \text{ miles.} \)
Hence \( h^2 \) is very small compared with \( hr \).
As a close approximation, we have then

\[
OT = \sqrt{2hr}.
\]

The dip

\[
= \angle TOQ
= 90^\circ - \angle COT = \angle OCT.
\]

Also,

\[
\tan OCT = \frac{OT}{CT} = \frac{\sqrt{2hr}}{r} = \sqrt{\frac{2h}{r}},
\]

so that, very approximately, we have

\[
\angle OCT = \sqrt{\frac{2h}{r}} \text{ radians}
\]

\[
= \left( \sqrt{\frac{2h}{r} \frac{180}{\pi}} \right)^\circ = \left[ \frac{180 \times 60 \times 60}{\pi} \sqrt{\frac{2h}{r}} \right]''.
\]

236. \textbf{Ex.} Taking the radius of the earth as 4000 miles, find the dip at the top of a lighthouse which is 264 feet above the sea, and the distance of the object.

Here \( r = 4000 \) miles, and \( h = 264 \) feet = \( \frac{1}{20} \) mile.

Hence \( h \) is very small compared with \( r \), so that

\[
OT = \sqrt{\frac{1}{10} \times 4000} = \sqrt{400} = 20 \text{ miles}.
\]

Also the dip = \( \sqrt{\frac{2h}{r}} \) radians = \( \frac{1}{200} \) radian

\[
= \left( \frac{1}{200} \times \frac{180 \times 60}{\pi} \right)'' = \left( \frac{51}{\pi} \right)'' = 17'11'' \text{ nearly}.
\]

\textbf{EXAMPLES. XLII.}

[\textit{Unless otherwise stated, the earth’s radius may be taken to be 4000 miles}]

1. Find in degrees, minutes, and seconds, the dip of the horizon from the top of a mountain 4200 feet high, the earth’s radius being \( 21 \times 10^6 \) feet.

2. The lighthouse is 196 feet high; how far off can it be seen?
3. If the radius of the earth be 4000 miles, find the height of a balloon when the dip is 1°.
   Find also the dip when the balloon is 2 miles high.

4. From the top of the mast of a ship, which is 66 feet above the sea, the light of a lighthouse which is known to be 132 feet high can just be seen; prove that its distance is 24 miles nearly.

5. From the top of a mast, 66 feet above the sea, the top of the mast of another ship can just be seen at a distance of 20 miles; prove that the heights of the masts are the same.

6. From the top of the mast of a ship which is 44 feet above the sea-level, the light of a lighthouse can just be seen; after sailing for 15 minutes the light can just be seen from the deck which is 11 feet above the sea-level; prove that the rate of sailing of the ship is nearly 16.33 miles per hour.

7. Prove that, if the height of the place of observation be $n$ feet, the distance that the observer can see is $\sqrt{\frac{3n}{2}}$ miles nearly.

8. There are 10 million metres in a quadrant of the earth's circumference. Find approximately the distance at which the top of the Eiffel tower should be visible, its height being 300 metres.

9. Three vertical posts are placed at intervals of a mile along a straight canal, each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point 8 inches below its top. Find the radius of the earth to the nearest mile.
CHAPTER XVIII.

INVERSE CIRCULAR FUNCTIONS.

237. If \( \sin \theta = a \), where \( a \) is a known quantity, we know, from Art. 82, that \( \theta \) is not definitely known. We only know that \( \theta \) is some one of a definite series of angles.

The symbol "\( \sin^{-1} a \)" is used to denote the smallest angle, whether positive or negative, that has \( a \) for its sine.

The symbol "\( \sin^{-1} a \)" is read in words as "sine minus one \( a \)," and must be carefully distinguished from \( \frac{1}{\sin a} \) which would be written, if so desired, in the form \( (\sin a)^{-1} \).

It will therefore be carefully noted that "\( \sin^{-1} a \)" is an angle, and denotes the smallest numerical angle whose sine is \( a \).

So "\( \cos^{-1} a \)" means the smallest numerical angle whose cosine is \( a \). Similarly "\( \tan^{-1} a \)," "\( \cot^{-1} a \)," "\( \cosec^{-1} a \)," "\( \sec^{-1} a \)," "\( \vers^{-1} a \)," and "\( \covers^{-1} a \)," are defined.

Hence \( \sin^{-1} a \) and \( \tan^{-1} a \) (and therefore cosec\(^{-1} a \) and cot\(^{-1} a \)) always lie between \(-90^\circ\) and \(+90^\circ\).

But \( \cos^{-1} a \) (and therefore sec\(^{-1} a \)) always lies between \(0^\circ\) and \(180^\circ\).

L. T.
238. The quantities \( \sin^{-1} a, \cos^{-1} a, \tan^{-1} a, \ldots \) are called Inverse Circular Functions.

The symbol \( \sin^{-1} a \) is often, especially in foreign mathematical books, written as "arc \( \sin a \); similarly \( \cos^{-1} a \) is written "arc \( \cos a, \)" and so for the other inverse ratios.

239. When \( a \) is positive, \( \sin^{-1} a \) clearly lies between \( 0^\circ \) and \( 90^\circ \); when \( a \) is negative, it lies between \(-90^\circ \) and \( 0^\circ \).

Ex. \( \sin^{-1} \frac{1}{2} = 30^\circ; \sin^{-1} \frac{-\sqrt{3}}{2} = -60^\circ. \)

When \( a \) is positive, there are two angles, one lying between \( 0^\circ \) and \( 90^\circ \) and the other lying between \(-90^\circ \) and \( 0^\circ \), each of which has its cosine equal to \( a \). [For example both \( 30^\circ \) and \(-30^\circ \) have their cosine equal to \( \frac{\sqrt{3}}{2} \).] In this case we take the smallest positive angle. Hence \( \cos^{-1} a \), when \( a \) is positive, lies between \( 0^\circ \) and \( 90^\circ \).

So \( \cos^{-1} a \), when \( a \) is negative, lies between \( 90^\circ \) and \( 180^\circ \).

Ex. \( \cos^{-1} \frac{1}{\sqrt{2}} = 45^\circ; \cos^{-1} \left( -\frac{1}{2} \right) = 120^\circ. \)

When \( a \) is positive, the angle \( \tan^{-1} a \) lies between \( 0^\circ \) and \( 90^\circ \); when \( a \) is negative, it lies between \(-90^\circ \) and \( 0^\circ \).

Ex. \( \tan^{-1} \sqrt{3} = 60^\circ; \tan^{-1} (-1) = -45^\circ. \)

240. Ex. 1. Prove that \( \sin^{-1} \frac{3}{5} - \cos^{-1} \frac{12}{13} = \sin^{-1} \frac{16}{65}. \)

Let \( \sin^{-1} \frac{3}{5} = a, \) so that \( \sin a = \frac{3}{5}, \)

and therefore \( \cos a = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}. \)
Let \( \cos^{-1} \frac{12}{13} = \beta \), so that \( \cos \beta = \frac{12}{13} \).

and therefore \( \sin \beta = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13} \).

Let \( \sin^{-1} \frac{16}{65} = \gamma \), so that \( \sin \gamma = \frac{16}{65} \).

We have then to prove that

\[ a - \beta = \gamma, \]

i.e. to shew that

\[ \sin (a - \beta) = \sin \gamma. \]

Now

\[ \sin (a - \beta) = \sin a \cos \beta - \cos a \sin \beta = \frac{3 \cdot 12}{5 \cdot 13} - \frac{4 \cdot 5}{5 \cdot 13} = \frac{36 - 20}{65} = \frac{16}{65} = \sin \gamma. \]

Hence the relation is proved.

**Ex. 2.** Prove that \( 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4} \).

Let \( \tan^{-1} \frac{1}{3} = a \), so that \( \tan a = \frac{1}{3} \),

and let \( \tan^{-1} \frac{1}{7} = \beta \), so that \( \tan \beta = \frac{1}{7} \).

We have then to shew that

\[ 2a + \beta = \frac{\pi}{4}. \]

Now

\[ \tan 2a = \frac{2 \tan a}{1 - \tan^2 a} = \frac{2 \cdot \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{6}{8} = \frac{3}{4}. \]

Also,

\[ \tan (2a + \beta) = \frac{\tan 2a + \tan \beta}{1 - \tan 2a \tan \beta} = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}} = \frac{21 + 4}{28 - 3} = \frac{25}{22} = \tan \frac{\pi}{4}. \]

\[ \therefore 2a + \beta = \frac{\pi}{4}. \]
Ex. 3. Prove that

\[4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}.\]

Let \( \tan^{-1} \frac{1}{5} = a, \) so that \( \tan a = \frac{1}{5}. \)

Then

\[\tan 2a = \frac{2 \tan a}{1 - \tan^2 a} = \frac{2}{1 - \frac{1}{25}} = \frac{5}{12},\]

and

\[\tan 4a = \frac{12}{1 - 25} = \frac{120}{119},\]

so that \( \tan 4a \) is nearly unity, and \( 4a \) therefore nearly \( \frac{\pi}{4}. \)

Let \( 4a = \frac{\pi}{4} + \tan^{-1} x. \)

\[\therefore \frac{120}{119} = \tan \left( \frac{\pi}{4} + \tan^{-1} x \right) = \frac{1 + x}{1 - x} \quad \text{(Art. 100)}.\]

\[\therefore x = \frac{1}{239}.\]

Hence

\[4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}.\]

Ex. 4. Prove that

\[\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a + b}{1 - ab}.\]

Let \( \tan^{-1} a = a, \) so that \( \tan a = a. \)

Let \( \tan^{-1} b = \beta, \) so that \( \tan \beta = b. \)

Also, let \( \tan^{-1} \left( \frac{a + b}{1 - ab} \right) = \gamma, \) so that \( \tan \gamma = \frac{a + b}{1 - ab}. \)

We have then to prove that \( a + \beta = \gamma. \)

Now

\[\tan (a + \beta) = \frac{\tan a + \tan \beta}{1 - \tan a \tan \beta} = \frac{a + b}{1 - ab} = \tan \gamma,\]

so that the relation is proved.
The above relation is merely the formula

\[ \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \]

expressed in inverse notation.

For put \( \tan x = a, \) so that \( x = \tan^{-1} a, \)
and \( \tan y = b, \) so that \( y = \tan^{-1} b. \)

Then

\[ \tan(x + y) = \frac{a + b}{1 - ab}. \]

\[ \therefore \ x + y = \tan^{-1} \frac{a + b}{1 - ab}, \]

i.e.

\[ \tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a + b}{1 - ab}. \]

In the above we have tacitly assumed that \( ab < 1, \) so that \( \frac{a + b}{1 - ab} \) is positive, and therefore \( \tan^{-1} \frac{a + b}{1 - ab} \) lies between 0° and 90°.

If, however, \( ab \) be > 1, then \( \frac{a + b}{1 - ab} \) and therefore according to our definition \( \tan^{-1} \frac{a + b}{1 - ab} \) is a negative angle. Here \( \gamma \) is therefore a negative angle and, since \( \tan(\pi + \gamma) = \tan \gamma, \) the formula should be

\[ \tan^{-1} a + \tan^{-1} b = \pi + \tan^{-1} \frac{a + b}{1 - ab}. \]

**Ex. 5.** Prove that

\[ \cos^{-1} \frac{3}{5} + 2 \tan^{-1} \frac{1}{5} = \sin^{-1} \frac{3}{5}. \]

Since \( 65^2 - 63^2 = 16^2, \) we have

\[ \cos^{-1} \frac{6}{5} = \tan^{-1} \frac{4}{3}. \]

Also, as in Ex. 1, \( \sin^{-1} \frac{4}{5} = \tan^{-1} \frac{4}{3}. \)

We have therefore to shew that

\[ \tan^{-1} \frac{4}{3} + 2 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{1}{2}. \]

Now

\[ \tan[2 \tan^{-1} \frac{1}{2}] = \frac{2 \tan[\tan^{-1} \frac{1}{2}]}{1 - \tan^2[\tan^{-1} \frac{1}{2}]} = \frac{2}{1 - \frac{1}{25}} = \frac{5}{2}, \]

so that

\[ 2 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{5}{2}. \]
Thus \[\tan \left( \tan^{-1} \frac{6}{3} + 2 \tan^{-1} \frac{1}{6} \right) = \tan \left( \tan^{-1} \frac{6}{9} + \tan^{-1} \frac{6}{12} \right)\]
\[= \frac{\frac{6}{3} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{6}{12}}{1 - \frac{6}{3} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{6}{12}} = \frac{\frac{1}{2} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{12}} = \frac{\frac{1}{12} + \frac{1}{12}}{1 - \frac{1}{12} \cdot \frac{1}{12} \cdot \frac{1}{12}} = \frac{\frac{1}{6}}{1 - \frac{1}{12}} = \frac{192 + 315}{706 - 80} = \frac{507}{626} = \frac{2}{3},\]

i.e. \[\tan^{-1} \frac{6}{3} + 2 \tan^{-1} \frac{1}{6} = \tan^{-1} \frac{2}{3}\]

\textbf{Ex. 6.} \textit{Solve the equation}

\[\tan^{-1} \frac{x + 1}{x - 1} + \tan^{-1} \frac{x - 1}{x} = \tan^{-1} (-7).\]

Taking the tangents of both sides of the equation, we have

\[\frac{\tan \left( \tan^{-1} \frac{x + 1}{x - 1} \right) + \tan \left( \tan^{-1} \frac{x - 1}{x} \right)}{1 - \tan \left( \tan^{-1} \frac{x + 1}{x - 1} \right) \tan \left( \tan^{-1} \frac{x - 1}{x} \right)} = \tan \{\tan^{-1} (-7)\}\]
\[= \tan (-7),\]
\[= -7,\]

i.e.

\[\frac{x + 1 + \frac{x - 1}{x}}{x - 1} = -7,\]
\[\frac{x + 1}{x - 1} + \frac{x - 1}{x} = -7,\]

i.e.

\[\frac{2x^2 - x + 1}{1 - x} = -7,\]

so that

\[x = 2.\]

This value makes the left-hand side of the given equation positive, so that there is no value of \(x\) strictly satisfying the given equation.

The value \(x = 2\) is a solution of the equation

\[\tan^{-1} \frac{x + 1}{x - 1} + \tan^{-1} \frac{x - 1}{x} = \pi + \tan^{-1} (-7).\]
INVERSE CIRCULAR FUNCTIONS.

EXAMPLES. XLIII.

[The student should verify the results of some of the following examples (e.g. Nos. 1—4, 8, 9, 12, 13) by an accurate graph.]

Prove that

1. \( \text{sin}^{-1} \frac{3}{5} + \text{sin}^{-1} \frac{8}{17} = \text{sin}^{-1} \frac{77}{85} \).

2. \( \text{sin}^{-1} \frac{5}{13} + \text{sin}^{-1} \frac{7}{25} = \text{cos}^{-1} \left( \frac{253}{325} \right) \).

3. \( \text{cos}^{-1} \frac{4}{5} + \text{tan}^{-1} \frac{3}{5} = \text{tan}^{-1} \frac{27}{11} \).

4. \( \text{cos}^{-1} \frac{4}{5} + \text{cos}^{-1} \frac{12}{13} = \text{cos}^{-1} \frac{33}{65} \).

5. \( \text{cos}^{-1} x = 2 \text{sin}^{-1} \sqrt{\frac{1-x}{2}} = 2 \text{cos}^{-1} \sqrt{\frac{1+x}{2}} \).

6. \( 2 \text{cos}^{-1} \frac{3}{\sqrt{13}} + \text{cot}^{-1} \frac{16}{63} + \frac{1}{2} \text{cos}^{-1} \frac{7}{25} = \pi \).

7. \( \text{tan}^{-1} \frac{1}{2} + \text{tan}^{-1} \frac{1}{3} = \text{sin}^{-1} \frac{1}{\sqrt{5}} + \text{cot}^{-1} 3 = 45^\circ \).

8. \( \text{tan}^{-1} \frac{1}{7} + \text{tan}^{-1} \frac{1}{13} = \text{tan}^{-1} \frac{2}{9} \).

9. \( \text{tan}^{-1} \frac{2}{3} = \frac{1}{2} \text{tan}^{-1} \frac{12}{5} \).

10. \( \text{tan}^{-1} \frac{1}{4} + \text{tan}^{-1} \frac{2}{9} = \frac{1}{2} \text{cos}^{-1} \frac{3}{5} \).

11. \( 2 \text{tan}^{-1} \frac{1}{5} + \text{tan}^{-1} \frac{1}{7} + 2 \text{tan}^{-1} \frac{1}{8} = \pi \).

12. \( \text{tan}^{-1} \frac{3}{4} + \text{tan}^{-1} \frac{3}{5} - \text{tan}^{-1} \frac{8}{19} = \pi \).

13. \( \text{tan}^{-1} \frac{1}{3} + \text{tan}^{-1} \frac{1}{5} + \text{tan}^{-1} \frac{1}{7} + \text{tan}^{-1} \frac{1}{8} = \pi \).

14. \( 3 \text{tan}^{-1} \frac{1}{4} + \text{tan}^{-1} \frac{1}{20} = \frac{\pi}{4} - \text{tan}^{-1} \frac{1}{1983} \).

15. \( 4 \text{tan}^{-1} \frac{1}{5} - \text{tan}^{-1} \frac{1}{70} + \text{tan}^{-1} \frac{1}{99} = \frac{\pi}{4} \).

16. \( \text{tan}^{-1} \frac{120}{119} = 2 \text{sin}^{-1} \frac{5}{13} \).

17. \( \text{tan}^{-1} \frac{m}{n} - \text{tan}^{-1} \frac{m-n}{m+n} = \frac{\pi}{4} \).

18. \( \text{tan}^{-1} t + \text{tan}^{-1} \frac{2t}{1-t^2} = \text{tan}^{-1} \frac{3t-t^3}{1-3t^2} \), \( t \) being positive,

if \( t < \frac{1}{\sqrt{3}} \) or \( > \sqrt{3} \), and \( = \pi + \text{tan}^{-1} \frac{3t-t^3}{1-3t^2} \) if \( t > \frac{1}{\sqrt{3}} \) and \( < \sqrt{3} \).
19. $\tan^{-1} \sqrt{\frac{a (a + b + c)}{bc}} + \tan^{-1} \sqrt{\frac{b (a + b + c)}{ca}} + \tan^{-1} \sqrt{\frac{c (a + b + c)}{ab}} = \pi.$

20. $\cot^{-1} \frac{ab + 1}{a - b} + \cot^{-1} \frac{bc + 1}{b - c} + \cot^{-1} \frac{ca + 1}{c - a} = 0.$

21. $\tan^{-1} n + \cot^{-1} (u + 1) = \tan^{-1} (u^2 + n + 1).$

22. $\cos \left( 2 \tan^{-1} \frac{1}{4} \right) - \sin \left( 4 \tan^{-1} \frac{1}{4} \right).$

23. $2 \tan^{-1} \left( \frac{\tan (45^\circ - \alpha) \tan \beta}{2} \right) = \cos^{-1} \left[ \frac{\sin 2\alpha + \cos \beta}{1 + \sin 2\alpha \cos \beta} \right].$

24. $\tan^{-1} x = 2 \tan^{-1} [\cosec \tan^{-1} x - \tan \cot^{-1} x].$

25. $2 \tan^{-1} \left[ \tan \frac{a}{2} \tan \left( \frac{\pi}{4} - \frac{\beta}{2} \right) \right] = \tan^{-1} \frac{\sin a \cos \beta}{\sin \beta + \cos \alpha}.$

26. Shew that

$$\cos^{-1} \sqrt{\frac{a-x}{a-b}} = \sin^{-1} \sqrt{\frac{x-b}{a-b}} = \cot^{-1} \sqrt{\frac{a-x}{x-b}} = \frac{1}{2} \sin^{-1} \frac{2\sqrt{(a-x)(x-b)}}{a-b}.$$ 

27. If $\cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b} = \alpha,$ prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab} \cos a + \frac{y^2}{b^2} = \sin^2 a.$$ 

Solve the equations

28. $\tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \beta.$

29. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}.$

30. $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}.$

31. $\tan^{-1} (x+1) + \cot^{-1} (x-1) = \sin^{-1} \frac{4}{5} + \cos^{-1} \frac{3}{5}.$

32. $\tan^{-1} (x+1) + \tan^{-1} (x-1) = \tan^{-1} \frac{8}{31}.$

33. $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \cosec x).$
34. \( \tan^{-1} x + 2 \cot^{-1} x = \frac{2}{3} \pi \). \\
35. \( \tan \cos^{-1} x = \sin \cot^{-1} \frac{1}{2} \).

36. \( \cot^{-1} x - \cot^{-1} (x + 2) = 15^\circ. \)

37. \( \cos^{-1} \frac{x^2 - 1}{x^2 + 1} + \tan^{-1} \frac{2x}{x^2 - 1} = \frac{2\pi}{3}. \)

38. \( \cot^{-1} x + \cot^{-1} (n^2 - x + 1) = \cot^{-1} (n - 1). \)

39. \( \sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}. \)

40. \( \sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}. \)

41. \( \tan^{-1} \frac{a}{x} + \tan^{-1} \frac{b}{x} + \tan^{-1} \frac{c}{x} + \tan^{-1} \frac{d}{x} = \pi. \)

42. \( \sec^{-1} \frac{x}{a} - \sec^{-1} \frac{x}{b} = \sec^{-1} b - \sec^{-1} a. \)

43. \( \cosec^{-1} x = \cosec^{-1} a + \cosec^{-1} b. \)

44. \( 2 \tan^{-1} \frac{x}{2} = \cos^{-1} \frac{1 - a^2}{1 + a^2} - \cos^{-1} \frac{1 - b^2}{1 + b^2}. \)

Draw the graphs of

45. \( \sin^{-1} x. \) [N.B. If \( y = \sin^{-1} x \), then \( x = \sin y \) and the graph bears the same relation to \( OY \) that the curve in Art. 62 bears to \( OX. \)]

46. \( \cos^{-1} x. \)

47. \( \tan^{-1} x. \)

48. \( \cot^{-1} x. \)

49. \( \cosec^{-1} x. \)

50. \( \sec^{-1} x. \)

51. By obtaining the intersections of the graphs of \( \tan x \) and \( 2x \), show that the least positive solution of the equation \( \tan^{-1} 2x = x \) is the circular measure of an angle of approximately \( 67^\circ. \)
CHAPTER XIX.

ON SOME SIMPLE TRIGONOMETRICAL SERIES.

241. To find the sum of the sines of a series of angles, the angles being in arithmetical progression.

Let the angles be

\[ \alpha, \alpha + \beta, \alpha + 2\beta, \ldots \{\alpha + (n - 1)\beta}\].

Let

\[ S = \sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) \ldots + \sin \{\alpha + (n - 1)\beta\} \]

By Art. 97 we have

\[ 2 \sin \alpha \sin \frac{\beta}{2} = \cos \left( \alpha - \frac{\beta}{2} \right) - \cos \left( \alpha + \frac{\beta}{2} \right), \]

\[ 2 \sin (\alpha + \beta) \sin \frac{\beta}{2} = \cos \left( \alpha + \frac{\beta}{2} \right) - \cos \left( \alpha + \frac{3\beta}{2} \right), \]

\[ 2 \sin (\alpha + 2\beta) \sin \frac{\beta}{2} = \cos \left( \alpha + \frac{3\beta}{2} \right) - \cos \left( \alpha + \frac{5\beta}{2} \right), \]

\[ \ldots \]

\[ 2 \sin \{\alpha + (n - 2)\beta\} \sin \frac{\beta}{2} = \cos \{\alpha + (n - \frac{5}{2})\beta\} - \cos \{\alpha + (n - \frac{3}{2})\beta\}, \]

and

\[ 2 \sin \{\alpha + (n - 1)\beta\} \sin \frac{\beta}{2} = \cos \{\alpha + (n - \frac{3}{2})\beta\} - \cos \{\alpha + (n - \frac{1}{2})\beta\} \]

By adding together these \( n \) lines, we have

\[ 2 \sin \frac{\beta}{2} \cdot S = \cos \left( \alpha - \frac{\beta}{2} \right) - \cos \{\alpha + (n - \frac{1}{2})\beta\}, \]
the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

\[ 2 \sin \frac{\beta}{2} \cdot S = 2 \sin \left\{ \alpha + \left( \frac{n-1}{2} \right) \beta \right\} \sin \frac{n\beta}{2}, \]

i.e.

\[ S = \frac{\sin \left\{ \alpha + \left( \frac{n-1}{2} \right) \beta \right\} \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}. \]

**Ex.** By putting \( \beta = 2\alpha \), we have

\[ \sin \alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin (2n-1)\alpha \]

\[ = \frac{\sin \left\{ \alpha + (n-1)\alpha \right\} \sin n\alpha}{\sin \alpha} = \frac{\sin^2 n\alpha}{\sin \alpha}. \]

242. To find the sum of the cosines of a series of angles, the angles being in arithmetical progression.

Let the angles be

\[ \alpha, \alpha + \beta, \alpha + 2\beta, \ldots \alpha + (n-1)\beta. \]

Let

\[ S = \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos [\alpha + (n-1)\beta]. \]

By Art. 97, we have

\[ 2 \cos \alpha \sin \frac{\beta}{2} = \sin \left( \alpha + \frac{\beta}{2} \right) - \sin \left( \alpha - \frac{\beta}{2} \right), \]

\[ 2 \cos (\alpha + \beta) \sin \frac{\beta}{2} = \sin \left( \alpha + \frac{3\beta}{2} \right) - \sin \left( \alpha + \frac{\beta}{2} \right), \]

\[ 2 \cos (\alpha + 2\beta) \sin \frac{\beta}{2} = \sin \left( \alpha + \frac{5\beta}{2} \right) - \sin \left( \alpha + \frac{3\beta}{2} \right), \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ 2 \cos \{ \alpha + (n-2)\beta \} \sin \frac{\beta}{2} = \sin \left[ \alpha + (n-\frac{3}{2})\beta \right] - \sin \left[ \alpha + (n-\frac{5}{2})\beta \right], \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ 2 \cos \{ \alpha + (n-2)\beta \} \sin \frac{\beta}{2} = \sin \left[ \alpha + (n-\frac{3}{2})\beta \right] - \sin \left[ \alpha + (n-\frac{5}{2})\beta \right], \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
and

\[ 2 \cos [a + (n-1)\beta] \sin \frac{\beta}{2} = \sin [a + (n-\frac{1}{2})\beta] - \sin [a + (n-\frac{3}{2})\beta]. \]

By adding together these \( n \) lines, we have

\[ 2S \times \sin \frac{\beta}{2} = \sin \{a + (n - \frac{1}{2})\beta\} - \sin \{a - \frac{\beta}{2}\}, \]

the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

\[ 2S \times \sin \frac{\beta}{2} = 2 \cos \left\{ a + \frac{n-1}{2} \beta \right\} \sin \frac{n\beta}{2}, \]

\[ i.e. \quad S = \frac{\cos \left\{ a + \frac{n-1}{2} \beta \right\} \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}. \]

243. Both the expressions for \( S \) in Arts. 241 and 242 vanish when \( \sin \frac{n\beta}{2} \) is zero, \( i.e. \) when \( \frac{n\beta}{2} \) is equal to any multiple of \( \pi \), \( i.e. \)

\[ \frac{n\beta}{2} = p\pi, \]

where \( p \) is any integer, \( i.e. \) when

\[ \beta = p \cdot \frac{2\pi}{n}. \]

Hence the sum of the sines (or cosines) of \( n \) angles, which are in arithmetical progression, vanishes when the common difference of the angles is any multiple of \( \frac{2\pi}{n} \).

Ex. \[ \cos a + \cos \left( a + \frac{2\pi}{n} \right) + \cos \left( a + \frac{4\pi}{n} \right) + ... \text{ to } n \text{ terms} = 0, \]
and \[ \sin a + \sin \left( a + \frac{4\pi}{n} \right) + \sin \left( a + \frac{8\pi}{n} \right) + \ldots \text{ to } n \text{ terms} = 0. \]

244. **Ex. 1.** Find the sum of
\[ \sin a - \sin (a + \beta) + \sin (a + 2\beta) - \ldots \text{ to } n \text{ terms}. \]
We have, by Art. 73,
\[ \sin (a + \beta + \pi) = -\sin (a + \beta), \]
\[ \sin (a + 2\beta + 2\pi) = \sin (a + 2\beta), \]
\[ \sin (a + 3\beta + 3\pi) = -\sin (a + 3\beta), \]

Hence the series
\[
= \sin a + \sin \left( a + \beta + \pi \right) + \sin \left( a + 2 \beta + \pi \right) + \ldots \\
+ \sin \left\{ a + \frac{n-1}{2} (\beta + \pi) \right\} \sin \frac{n (\beta + \pi)}{2}, \text{ by Art. 241,} \\
= \sin a + \sin \left( a + \frac{n-1}{2} (\beta + \pi) \right) \sin \frac{n (\beta + \pi)}{2}.
\]

**Ex. 2.** Find the sum of the series
\[ \cos^4 a + \cos^3 2a + \cos^3 3a + \ldots \text{ to } n \text{ terms}. \]
By Art. 107, we have
\[ \cos 3a = 4 \cos^3 a - 3 \cos a, \]
so that
\[ 4 \cos^3 a = 3 \cos a + \cos 3a. \]
So
\[ 4 \cos^3 2a = 3 \cos 2a + \cos 6a, \]
\[ 4 \cos^3 3a = 3 \cos 3a + \cos 9a, \]

Hence, if \( S \) be the given series, we have
\[ 4S = (3 \cos a + \cos 3a) + (3 \cos 2a + \cos 6a) + (3 \cos 3a + \cos 9a) + \ldots \]
\[ = 3 \left( \cos a + \cos 2a + \cos 3a + \ldots \right) + (\cos 3a + \cos 6a + \cos 9a + \ldots) \]
\[ = 3 \frac{\sin \frac{n-1}{2} a \sin \frac{n a}{2}}{\sin \frac{a}{2}} + \frac{\cos \frac{3a + n-1}{2} 3a \sin \frac{n a}{2}}{\sin \frac{3a}{2}} \]
\[ = 3 \frac{\cos \frac{n+1}{2} a \sin \frac{n a}{2}}{\sin \frac{a}{2}} + \frac{\cos \frac{3(n+1)}{2} a \sin \frac{3n a}{2}}{\sin \frac{3a}{2}}. \]
In a similar manner we can obtain the sum of the cubes of the sines of a series of angles in A.P.

Cor. Since

\[ 2 \sin^2 a = 1 - \cos 2a, \quad \text{and} \quad 2 \cos^2 a = 1 + \cos 2a, \]

we can obtain the sum of the squares.

Since again

\[ 8 \sin^4 a = 2 [1 - \cos 2a]^2 \]
\[ = 2 - 4 \cos 2a + 2 \cos^2 2a = 3 - 4 \cos 2a + \cos 4a, \]

we can obtain the sum of the 4th powers of the sines. Similarly for the cosines.

**Ex. 3.** Sum to \( n \) terms the series

\[ \cos a \sin \beta + \cos 3a \sin 2\beta + \cos 5a \sin 3\beta + \ldots \] to \( n \) terms.

Let \( S \) denote the series.

Then

\[ 2S = \{ \sin (a + \beta) - \sin (a - \beta) \} + \{ \sin (3a + 2\beta) - \sin (3a - 2\beta) \} + \ldots \]
\[ = \{ \sin (a + \beta) + \sin (3a + 2\beta) + \sin (5a + 3\beta) + \ldots \} \]
\[ - \{ \sin (a - \beta) + \sin (3a - 2\beta) + \sin (5a - 3\beta) + \ldots \} \]

\[ \frac{\sin \left\{ (a + \beta) + \frac{n - 1}{2} (2a + \beta) \right\} \sin n \frac{2a + \beta}{2}}{\sin \frac{2a + \beta}{2}} \]

\[ - \frac{\sin \left\{ (a - \beta) + \frac{n - 1}{2} (2a - \beta) \right\} \sin n \frac{2a - \beta}{2}}{\sin \frac{2a - \beta}{2}}, \text{ by Art. 241,} \]

\[ \frac{\sin \left\{ na + \frac{n + 1}{2} \beta \right\} \sin n \frac{(2a + \beta)}{2}}{\sin \frac{2a + \beta}{2}} \]
\[ - \frac{\sin \left\{ na - \frac{n + 1}{2} \beta \right\} \sin n \frac{(2a - \beta)}{2}}{\sin \frac{2a - \beta}{2}}. \]
**Ex. 4.** \( A_1A_2...A_n \) is a regular polygon of \( n \) sides inscribed in a circle, whose centre is \( O \), and \( P \) is any point on the arc \( A_nA_1 \) such that the angle \( POA_1 \) is \( \theta \); find the sum of the lengths of the lines joining \( P \) to the angular points of the polygon.

Each of the angles \( A_1OA_2, A_2OA_3, ...A_nOA_1 \) is \( \frac{2\pi}{n} \), so that the angles \( POA_1, POA_2, ... \) are respectively

\[
\theta, \quad \theta + \frac{2\pi}{n}, \quad \theta + \frac{4\pi}{n}, ...
\]

Hence, if \( r \) be the radius of the circle, we have

\[
PA_1 = 2r \sin \frac{POA_1}{2} = 2r \sin \frac{\theta}{2},
\]

\[
PA_2 = 2r \sin \frac{POA_2}{2} = 2r \sin \left( \frac{\theta + \frac{\pi}{n}}{2} \right),
\]

\[
PA_3 = 2r \sin \frac{POA_3}{2} = 2r \sin \left( \frac{\theta + \frac{2\pi}{n}}{2} \right),
\]

... ........................................

Hence the required sum

\[
= 2r \left[ \sin \frac{\theta}{2} + \sin \left( \frac{\theta}{2} + \frac{\pi}{n} \right) + \sin \left( \frac{\theta}{2} + \frac{2\pi}{n} \right) + ... \right. \text{to } n \text{ terms}
\]

\[
= 2r \frac{\sin \frac{\theta + \pi - \pi/n}{2} \sin \frac{n \pi}{2n}}{\sin \frac{\pi}{2n}} \quad \text{(Art. 241)}
\]

\[
= 2r \cosec \frac{\pi}{2n} \cdot \sin \left[ \frac{\pi}{2} + \frac{\theta}{2} - \frac{\pi}{2n} \right]
\]

\[
= 2r \cosec \frac{\pi}{2n} \cos \left( \frac{\theta}{2} - \frac{\pi}{2n} \right).
\]

**EXAMPLES. XLIV.**

Sum the series:

1. \( \cos \theta + \cos 3\theta + \cos 5\theta + ... \) to \( n \) terms.

2. \( \cos \frac{A}{2} + \cos 2A + \cos \frac{7A}{2} + ... \) to \( n \) terms.

Prove that

3. \[
\frac{\sin a + \sin 2a + \sin 3a + ... + \sin na}{\cos a + \cos 2a + ... + \cos na} = \tan \frac{n + 1}{2} a.
\]
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[Exs. XLIV.]

4. \[
\frac{\sin a + \sin 3a + \sin 5a + \ldots + \sin (2n-1)a}{\cos a + \cos 3a + \cos 5a + \ldots + \cos (2n-1)a} = \tan na.
\]

5. \[
\frac{\sin a - \sin (a+\beta) + \sin (a+2\beta) + \ldots}{\cos a - \cos (a+\beta) + \cos (a+2\beta) + \ldots} = \tan \left\{ a + \frac{n-1}{2} (\pi + \beta) \right\}.
\]

Sum the following series:

6. \[
\cos \frac{\pi}{2n+1} + \cos \frac{3\pi}{2n+1} + \cos \frac{5\pi}{2n+1} + \ldots \text{ to } n \text{ terms.}
\]

7. \[
\cos a - \cos (a+\beta) + \cos (a+2\beta) - \ldots \text{ to } 2n \text{ terms.}
\]

8. \[
\sin \theta + \sin \frac{n-4}{n-2} \theta + \sin \frac{n-6}{n-2} \theta + \ldots \text{ to } n \text{ terms.}
\]

9. \[
\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots + \sin (4n-1)x.
\]

10. \[
\sin a \sin 2a + \sin 2a \sin 3a + \sin 3a \sin 4a + \ldots \text{ to } n \text{ terms.}
\]

11. \[
\cos a \sin 2a + \sin 2a \cos 3a + \cos 3a \sin 4a + \sin 4a \cos 5a + \ldots \text{ to } 2n \text{ terms.}
\]

12. \[
\sin a \sin 3a + \sin 2a \sin 4a + \sin 3a \sin 5a + \ldots \text{ to } n \text{ terms.}
\]

13. \[
\cos a \cos \beta + \cos 3a \cos 2\beta + \cos 5a \cos 3\beta + \ldots \text{ to } n \text{ terms.}
\]

14. \[
\sin^2 a + \sin^2 2a + \sin^2 3a + \ldots \text{ to } n \text{ terms.}
\]

15. \[
\sin^2 \theta + \sin^2 (\theta + a) + \sin^2 (\theta + 2a) + \ldots \text{ to } n \text{ terms.}
\]

16. \[
\sin^3 a + \sin^3 2a + \sin^3 3a + \ldots \text{ to } n \text{ terms.}
\]

17. \[
\sin^4 a + \sin^4 2a + \sin^4 3a + \ldots \text{ to } n \text{ terms.}
\]

18. \[
\cos^4 a + \cos^4 2a + \cos^4 3a + \ldots \text{ to } n \text{ terms.}
\]

19. \[
\cos \theta \cos 2\theta \cos 3\theta + \cos 2\theta \cos 3\theta \cos 4\theta + \ldots \text{ to } n \text{ terms.}
\]

20. \[
\sin a \sin (a+\beta) - \sin (a+\beta) \sin (a+2\beta) + \ldots \text{ to } 2n \text{ terms.}
\]

21. From the sum of the series

\[
\sin a + \sin 2a + \sin 3a + \ldots \text{ to } n \text{ terms},
\]
deduce (by making \(a\) very small) the sum of the series

\[
1 + 2 + 3 + \ldots + n.
\]

22. From the result of the example of Art. 211 deduce the sum of

\[
1 + 3 + 5 + \ldots \text{ to } n \text{ terms.}
\]

23. If \(a = \frac{2\pi}{17}\),
prove that
\[
2 (\cos a + \cos 2a + \cos 4a + \cos 8a)
\]
and
\[
2 (\cos 3a + \cos 5a + \cos 6a + \cos 7a)
\]
are the roots of the equation
\[
x^2 + x - 4 = 0.
\]
24. \( ABCD \ldots \) is a regular polygon of \( n \) sides which is inscribed in a circle, whose centre is \( O \) and whose radius is \( r \), and \( P \) is any point on the arc \( AB \) such that \( POA \) is \( \theta \). Prove that

\[
PA \cdot PB + PA \cdot PC + PA \cdot PD + \ldots + PB \cdot PC + \ldots
= r^2 \left[ 2 \cos^2 \left( \frac{\theta}{2} - \frac{\pi}{2n} \right) \csc^2 \frac{\pi}{2n} - n \right].
\]

25. Two regular polygons, each of \( n \) sides, are circumscribed to and inscribed in a given circle. If an angular point of one of them be joined to each of the angular points of the other, then the sum of the squares of the straight lines so drawn is to the sum of the areas of the polygons as

\[
2 : \sin \frac{2\pi}{n}.
\]

26. \( A_1, A_2, \ldots A_{2n+1} \) are the angular points of a regular polygon inscribed in a circle, and \( O \) is any point on the circumference between \( A_1 \) and \( A_{2n+1} \); prove that

\[
OA_1 + OA_3 + \ldots + OA_{2n+1} = OA_2 + OA_4 + \ldots + OA_{2n}.
\]

27. If perpendiculars be drawn on the sides of a regular polygon of \( n \) sides from any point on the inscribed circle whose radius is \( a \), prove that

\[
\frac{2}{n} \sum \left( \frac{p}{a} \right)^2 = 3, \text{ and } \frac{2}{n} \sum \left( \frac{p}{a} \right)^3 = 5.
\]
CHAPTER XX.

ELIMINATION.

245. It sometimes happens that we have two equations each containing one unknown quantity. In this case there must clearly be a relation between the constants of the equations in order that the same value of the unknown quantity may satisfy both. For example, suppose we knew that an unknown quantity \( x \) satisfied both of the equations

\[
ax + b = 0 \text{ and } cx^2 + dx + e = 0.
\]

From the first equation, we have

\[
x = -\frac{b}{a},
\]

and this satisfies the second, if

\[
c\left(-\frac{b}{a}\right)^2 + d\left(-\frac{b}{a}\right) + e = 0,
\]

i.e. if

\[
lc - abd + a^2e = 0.
\]

This latter equation is the result of eliminating \( x \) between the above two equations, and is often called their eliminant.
246. Again, suppose we knew that an angle \( \theta \) satisfied both of the equations

\[
\sin^3 \theta = b, \quad \text{and} \quad \cos^3 \theta = c,
\]

so that \( \sin \theta = b^{\frac{1}{3}}, \) and \( \cos \theta = c^{\frac{1}{3}}. \)

Now we always have, for all values of \( \theta, \)

\[
\sin^2 \theta + \cos^2 \theta = 1,
\]

so that in this case \( b^{\frac{2}{3}} + c^{\frac{2}{3}} = 1. \)

This is the result of eliminating \( \theta. \)

247. Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice, a considerable amount of artifice and ingenuity is often required in seemingly simple cases.

So, between any three equations involving two unknown quantities, we can theoretically eliminate both of the unknown quantities.

248. Some examples of elimination are appended.

Ex. 1. Eliminate \( \theta \) from the equations

\[
a \cos \theta + b \sin \theta = c,
\]

and

\[
d \cos \theta + e \sin \theta = f.
\]

Solving for \( \cos \theta \) and \( \sin \theta \) by cross multiplication, or otherwise, we have

\[
\frac{\cos \theta}{bf - ce} = \frac{\sin \theta}{cd - af} = \frac{1}{bd - ae}.
\]

\[
\therefore 1 = \cos^2 \theta + \sin^2 \theta = \frac{(bf - ce)^2 + (cd - af)^2}{(bd - ae)^2},
\]

so that \( (bf - ce)^2 + (cd - af)^2 = (bd - ae)^2. \)
**Ex. 2. Eliminate \( \theta \) between**

\[
\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad \ldots \ldots \ldots \ldots (1),
\]

and

\[
\frac{ax \sin \theta}{\cos^2 \theta} + \frac{by \cos \theta}{\sin^2 \theta} = 0 \quad \ldots \ldots \ldots \ldots (2).
\]

From (2) we have \( ax \sin^3 \theta = -by \cos^3 \theta \).

\[
\therefore \quad \frac{\sin \theta}{-(by)^{\frac{3}{2}}} = \frac{\cos \theta}{(ax)^{\frac{1}{2}}} = \frac{\sqrt{\sin^2 \theta + \cos^2 \theta}}{\sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}}}
\]

(Todhunter and Loney's *Algebra for Beginners*, Art. 371)

\[
= \frac{1}{\sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}}}.
\]

Hence

\[
\frac{1}{\sin \theta} = -\frac{\sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}}}{(by)^{\frac{3}{2}}},
\]

and

\[
\frac{1}{\cos \theta} = \frac{\sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}}}{(ax)^{\frac{3}{2}}},
\]

so that (1) becomes

\[
a^2 - b^2 = \sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}} \left[ ax \cdot \frac{1}{(ax)^{\frac{1}{2}}} - by \left\{ -\frac{1}{(by)^{\frac{3}{2}}} \right\} \right]
\]

\[
= \sqrt{(by)^{\frac{3}{2}} + (ax)^{\frac{3}{2}}} \{ (ax)^{\frac{3}{2}} + (by)^{\frac{3}{2}} \}
\]

\[
= \{ (ax)^{\frac{3}{2}} + (by)^{\frac{3}{2}} \} \frac{3}{2},
\]

i.e.

\[
(ax)^{\frac{3}{2}} + (by)^{\frac{3}{2}} = (a^2 - b^2)^{\frac{3}{2}}.
\]

The student who shall afterwards become acquainted with Analytic Geometry will find that the above is the solution of an important problem concerning normals to an ellipse.

**Ex. 3. Eliminate \( \theta \) from the equations**

\[
\frac{x}{a} \cos \theta - \frac{y}{b} \sin \theta = \cos 2\theta \quad \ldots \ldots \ldots \ldots (1),
\]

and

\[
\frac{x}{a} \sin \theta + \frac{y}{b} \cos \theta = 2 \sin 2\theta \quad \ldots \ldots \ldots \ldots (2).
\]
Multiplying (1) by \( \cos \theta \), (2) by \( \sin \theta \), and adding, we have

\[
\frac{x}{a} = \cos \theta \cos 2\theta + 2 \sin \theta \sin 2\theta
\]

\[
= \cos \theta + \sin \theta \sin 2\theta = \cos \theta + 2 \sin^2 \theta \cos \theta \quad \ldots \ldots \ldots \ldots \ldots \quad (3).
\]

Multiplying (2) by \( \cos \theta \), (1) by \( \sin \theta \), and subtracting, we have

\[
\frac{y}{b} = 2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta
\]

\[
= \sin 2\theta \cos \theta + \sin \theta = \sin \theta + 2 \sin \theta \cos^2 \theta \quad \ldots \ldots \ldots \ldots \ldots \quad (4).
\]

Adding (3) and (4), we have

\[
\frac{x}{a} + \frac{y}{b} = (\sin \theta + \cos \theta) [1 + 2 \sin \theta \cos \theta]
\]

\[
= (\sin \theta + \cos \theta) [\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta]
\]

\[
= (\sin \theta + \cos \theta)^3,
\]

that

\[
\sin \theta + \cos \theta = \left( \frac{x}{a} + \frac{y}{b} \right)^{\frac{1}{3}} \quad \ldots \ldots \ldots \ldots \ldots \quad (5).
\]

Subtracting (4) from (3), we have

\[
\frac{x}{a} - \frac{y}{b} = (\cos \theta - \sin \theta) (1 - 2 \sin \theta \cos \theta)
\]

\[
= (\cos \theta - \sin \theta)^3,
\]

so that

\[
\cos \theta - \sin \theta = \left( \frac{x}{a} - \frac{y}{b} \right)^{\frac{1}{3}} \quad \ldots \ldots \ldots \ldots \ldots \quad (6).
\]

Squaring and adding (5) and (6), we have

\[
2 = \left( \frac{x}{a} + \frac{y}{b} \right)^{\frac{2}{3}} + \left( \frac{x}{a} - \frac{y}{b} \right)^{\frac{2}{3}}.
\]

**EXAMPLES. XLV.**

Eliminate \( \theta \) from the equations

1. \( a \cos \theta + b \sin \theta = c \), and \( b \cos \theta - a \sin \theta = d \).

2. \( x = a \cos (\theta - \alpha) \), and \( y = b \cos (\theta - \beta) \).

3. \( a \cos 2\theta = b \sin \theta \), and \( c \sin 2\theta = d \cos \theta \).

4. \( a \sin \alpha - b \cos a = 2b \sin \theta \), and \( a \sin 2\alpha - b \cos 2\theta = a \).

5. \( x \sin \theta - y \cos \theta = \sqrt{x^2 + y^2} \), and \( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} = \frac{1}{x^2 + y^2} \).
6. \[
\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1,
\]
and \[x \sin \theta - y \cos \theta = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}.
\]\n
7. \[\sin \theta - \cos \theta = p, \text{ and } \cos \theta - \sin \theta = q.
\]

8. \[x = a \cos \theta + b \cos 2\theta, \text{ and } y = a \sin \theta + b \sin 2\theta.
\]

9. If \[m = \cosec \theta - \sin \theta, \text{ and } n = \sec \theta - \cos \theta,
\]
prove that \[m^2 + n^2 = (mn) - \frac{1}{3}.
\]

10. Prove that the result of eliminating \(\theta\) from the equations
\[x \cos (\theta + a) + y \sin (\theta + a) = a \sin 2\theta,
\]
and \[y \cos (\theta + a) - x \sin (\theta + a) = 2a \cos 2\theta,
\]
is \[(x \cos a + y \sin a)^2 + (x \sin a - y \cos a)^2 = (2a)^2.
\]

Eliminate \(\theta\) and \(\phi\) from the equations

11. \[\sin \theta + \sin \phi = a, \cos \theta + \cos \phi = b, \text{ and } \theta - \phi = a.
\]

12. \[-\tan \theta + \tan \phi = x, \cos \theta + \cos \phi = y, \text{ and } \theta + \phi = a.
\]

13. \[a \cos^2 \theta + b \sin^2 \theta = c, \ b \cos^2 \phi + a \sin^2 \phi = d,
\]
and \[a \tan \theta = b \tan \phi.
\]

14. \[\cos \theta + \cos \phi = a, \cot \theta + \cot \phi = b, \text{ and } \cosec \theta + \cosec \phi = c.
\]

15. \[a \sin \theta = b \sin \phi, \ a \cos \theta + b \cos \phi = c, \text{ and } x = y \tan (\theta + \phi).
\]

16. \[\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1, \ \frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,
\]
and \[a^2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} + b^2 \cos \frac{\theta}{2} \cos \frac{\phi}{2} = c^2.
\]
CHAPTER XXI.

PROJECTIONS.

249. Let $PQ$ be any straight line, and from its ends, $P$ and $Q$, let perpendiculars be drawn to a fixed straight line $OA$. Then $MN$ is called the projection of $PQ$ on $OA$. 
If $MN$ be in the same direction as $OX$, it is positive; if in the opposite direction, it is negative.

250. If $\theta$ be the angle between any straight line $PQ$ and a fixed line $OA$, the projection of $PQ$ on $OA$ is $PQ \cos \theta$.

Whatever be the direction of $PQ$ draw, through $P$, a straight line $PL$ parallel to $OA$ and let it and $QN$, both produced if necessary, meet in $R$.

Then, in each figure, the angle $LPQ$ or the angle $AUQ$ is equal to $\theta$.

Also $MN = PR = PQ \cos LPQ = PQ \cos \theta$,

by the definitions of Art. 50.

Similarly, the projection of $PQ$ on a line perpendicular to $OA = RQ$

$= PQ \sin LPQ = PQ \sin \theta$.

The projections of any line $PQ$ on a line to which $PQ$ is inclined at any angle $\theta$, and on a perpendicular line, are therefore $PQ \cos \theta$ and $PQ \sin \theta$.

251. We might therefore, in Art. 50, have defined the cosine as the ratio to $OP$ of the projection of $OP$ on the initial line, and, similarly, the sine as the ratio to $OP$ of the projection of $OP$ on a line perpendicular to the initial line.

This method of looking upon the definition of the cosine and sine is often useful.

252. The projection of $PQ$ upon the fixed line $OA$ is equal to the sum of the projections on $OA$ of any broken line beginning at $P$ and ending at $Q$. 
Let $PEFGQ$ be any broken line joining $P$ and $Q$. Draw $PM$, $QN$, $ER$, $FS$, and $GT$ perpendicular to $OA$.

The projection of $PE$ is $MR$ and is positive.  
The projection of $EF$ is $RS$ and is negative.  
The projection of $FG$ is $ST$ and is positive.  
The projection of $GQ$ is $TN$ and is negative.  
The sum of the projections of the broken line $PEFGQ$ therefore

\[= MR + RS + ST + TN\]
\[= MR - SR + ST - NT\]
\[= MS + SN\]
\[= MN.\]

A similar proof will hold whatever be the positions of $P$ and $Q$, and however broken the lines joining them may be.

**Cor.** The sum of the projections of any broken line, joining $P$ to $Q$, is equal to the sum of the projections of any other broken line joining the same two points; for each sum is equal to the projection of the straight line $PQ$.

**253. General Proofs, by Projections, of the Addition and Subtraction Theorems.**

Let $AOB$ be the angle $A$ and $BOC$ the angle $B$. On
\( OC \), the bounding line of the angle \( A + B \), take any point \( P \), and draw \( PN \) perpendicular to \( OB \) and produce it to meet \( OA \) in \( L \).

Then \( \angle ALP = \angle LNO + \angle AOB = 90^\circ + A \).

(i) To prove \( \cos (A + B) = \cos A \cos B - \sin A \sin B \).

\[
OP \cdot \cos (A + B) = OP \cos AOP
\]

= projection of \( OP \) on \( OA \) \hspace{1cm} \text{(Art. 250)}

= projection of \( ON \) on \( OA \) + projection of \( NP \) on \( OA \)

\[
= ON \cos AON + NP \cos ALP
\]

\[
= OP \cos B \cdot \cos A + OP \sin B \cdot \cos (90^\circ + A)
\]

\[
= OP (\cos A \cos B - \sin A \sin B).
\]

Hence the result (i), on division by \( OP \).

(ii) To prove \( \sin (A + B) = \sin A \cos B + \cos A \sin B \).

\[
OP \cdot \sin (A + B) = OP \cdot \sin AOP
\]

= projection of \( OP \) on a perpendicular to \( OA \) \hspace{1cm} \text{(Art. 250)}

= sum of projections of \( ON, NP \) on the perp. to \( OA \) \hspace{1cm} \text{(Art. 252)}

\[
= ON \sin A + NP \sin ALP
\]
\[= OP \cos B \cdot \sin A + OP \sin B \cdot \sin (90^\circ + A) \quad \text{(Art. 250)}\]
\[= OP [\sin A \cos B + \cos A \sin B]. \quad \text{(Art. 70).}\]

Hence the result (ii).

The above proof holds, as in the subjoined figures, for all positions of the bounding lines \(OB\) and \(OC\).

254. In the case of the subtraction theorem, let \(\angle AOB\) be the angle \(A\), and let the angle \(BOC\) be equal to \(B\) described negatively, so that \(AOC\) is the angle \(A - B\); also \(OC\) is inclined to \(OB\) at an angle which, with the proper sign prefixed, is \(-B\).

In \(OC\), the bounding line of the angle we are considering, take any point \(P\); draw \(PN\) perpendicular to \(OB\) and produce it to meet \(OA\) in \(L\).
To prove \( \cos (A - B) = \cos A \cos B + \sin A \sin B \).

\[ OP \cos (A - B) = OP \cos A \angle OC \]  
projection of \( OP \) on \( OA \)  
(Art. 250)

= projection of \( ON \) on \( OA \) + projection of \( NP \) on \( OA \)  
(Art. 252)

= \( ON \cos A + NP \cos (90^\circ + A) \)  
(Art. 250)

= \( OP \cos (-B) \cos A + OP \sin (-B) \cdot \cos (90^\circ + A) \)  
(Art. 250)

= \( OP \cos B \cos A + OP (-\sin B)(-\sin A) \)  
(Art. 68, 70)

= \( OP [\cos A \cos B + \sin A \sin B] \).

Hence \( \cos (A - B) = \cos A \cos B + \sin A \sin B \).

To prove \( \sin (A - B) = \sin A \cos B - \cos A \sin B \).

\[ OP \sin (A - B) = OP \cdot \sin A \angle OC \]  
projection of \( OP \) on a perpendicular to \( OA \)  
(Art. 250)

= sum of the projections of \( ON, NP \) on the perpendicular to \( OA \)  
(Art. 252)

= \( ON \sin A + NP \sin (90^\circ + A) \)  
(Art. 250)

= \( OP \cos (-B) \cdot \sin A + OP \sin (-B) \cdot \sin (90^\circ + A) \)  
(Art. 250)

= \( OP \cos B \sin A - OP \sin B \cos A \).  
(Art. 68, 70)

\[ \cdot \cdot \cdot \sin (A - B) = \sin A \cos B - \cos A \sin B. \]

These proofs hold whatever be the positions of the

bounding lines \( OB \) and \( OC \), as, for example, in the subjoined figure.
MISCELLANEOUS EXAMPLES.

1. Show that if an angle $a$ be divided into two parts, so that the ratio of the tangents of the parts is $\lambda$, the difference $x$ between the parts is given by

$$\sin x = \frac{\lambda - 1}{\lambda + 1} \sin a.$$  

2. If $\tan (\pi \cos \theta) = \cot (\pi \sin \theta)$, prove that

$$\cos \left( \theta - \frac{\pi}{4} \right) = \frac{1}{2\sqrt{2}}.$$  

3. In any triangle $ABC$, show that

$$\frac{a - b}{c} = \frac{\tan \frac{A}{2} - \tan \frac{B}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}}, \quad \text{and} \quad \frac{a + b}{c} = \frac{1 + \tan \frac{A}{2} \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}}.$$  

4. An aeroplane is travelling east with a constant velocity of 18 miles per hour at a constant height above the ground. At a certain time a man observes it due north of him at an angle of elevation of $90^\circ 30'$. At the end of one minute he sees it in a direction $62^\circ$ east of north. At what height is the aeroplane travelling, and what is the angle of elevation at which the man sees it in the second observed position?

5. If the sides of a triangle are 51, 35 and 26 feet, find the sides of a triangle, on a base of 41 feet, which shall have the same area and perimeter as the first.

6. Prove that

$$\sin \cot^{-1} \cos \tan^{-1} x = \sqrt{\frac{x^2 + 1}{x^2 + 2}}.$$  

7. Eliminate $\theta$ from the equations

$$\sin (\theta + a) = a, \quad \cos^2 (\theta + \beta) = b.$$  

8. Show that, whatever be the value of $\theta$, the expression

$$a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$$

lies between

$$\frac{a+c}{2} + \frac{1}{2} \sqrt{b^2 + (a-c)^2} \quad \text{and} \quad \frac{a+c}{2} - \frac{1}{2} \sqrt{b^2 + (a-c)^2}.$$  

9. If

$$\sin x = k \sin (A - x),$$

show that

$$\tan \left( x - \frac{A}{2} \right) = \frac{k - 1}{k + 1} \tan \frac{A}{2},$$

and, by means of the Tables, solve the equation when $k = 3$ and $A = 50^\circ$.

10. Express

$$\tan \theta + \tan \left( \theta + \frac{\pi}{3} \right) + \tan \left( \theta + \frac{2\pi}{3} \right)$$

in terms of $\tan 3\theta$.

Hence, or otherwise, solve the equation

$$\tan \theta + \tan \left( \theta + \frac{\pi}{3} \right) + \tan \left( \theta + \frac{2\pi}{3} \right) = 3.$$

11. In a triangle $ABC$, if $\tan \frac{A}{2}, \tan \frac{B}{2}$, and $\tan \frac{C}{2}$ are in arithmetic progression, then $\cos A$, $\cos B$, and $\cos C$ are also in arithmetic progression.

12. A man standing on the sea shore observes two buoys in the same direction, the line through them making an angle $a$ with the shore. He then walks along the shore a distance $a$, when he finds the buoys subtend an angle $a$ at his eye; and on walking a further distance $b$ he finds that they again subtend an angle $a$ at his eye. Show that the distance between the buoys is $\left( a + \frac{b}{2} \right) \sec a - \frac{2a(a+b)}{2a+b} \cos a$, assuming the shore to be straight, and neglecting the height of the man's eye above the sea.

13. The bisectors of the angles of a triangle $ABC$ meet its circum-circle in the points $D, E, F$ respectively. Show that the area of the triangle $DEF$ is to that of $ABC$ as $R:2r$.

14. The alternate angles of a regular pentagon are joined forming another regular pentagon; find the ratio of the areas of the two pentagons.
15. If \( \phi = \tan^{-1} \frac{x^{\sqrt{3}}}{2k-x} \), and \( \theta = \tan^{-1} \frac{2x-k}{k^{\sqrt{3}}} \), prove that one value of \( \phi - \theta \) is \( 30^\circ \).

16. If

\[
m^2 + m'^2 + 2mm' \cos \theta = 1,
\]

\[
n^2 + n'^2 + 2nn' \cos \theta = 1,
\]

and

\[
nm + mn' + (mn' + m'n) \cos \theta = 0,
\]

prove that

\[
m^2 + n^2 = \csc^2 \theta.
\]

17. If \( x \) be real, prove that

\[
\frac{x^2 - 2x \cos a + 1}{x^2 - 2x \cos b + 1} \text{ lies between } \frac{\sin^2 \frac{a}{2}}{\sin^2 \frac{b}{2}} \text{ and } \frac{\cos^2 \frac{a}{2}}{\cos^2 \frac{b}{2}}.
\]

18. Prove that the area of a circle exceeds the area of a regular polygon of \( n \) sides and of equal perimeter in the ratio of

\[
\tan \frac{\pi}{n} : \frac{\pi}{n}.
\]

19. If \( \frac{\sin (2a-\theta)}{\sin \theta} = 1 + x \), where \( x \) is very small, show that

\[
\frac{\cos \theta}{\cos a} = 1 + \frac{1}{2} x \tan^2 a, \text{ approximately}.
\]

20. If

\[
2\sigma = a + \beta + \gamma + \delta,
\]

prove that

\[
\cos (\sigma - a) \cos (\sigma - \beta) \cos (\sigma - \gamma) \cos (\sigma - \delta) + \sin (\sigma - a) \sin (\sigma - \beta) \sin (\sigma - \gamma) \sin (\sigma - \delta) = \cos a \cos \beta \cos \gamma \cos \delta + \sin a \sin \beta \sin \gamma \sin \delta.
\]

21. Solve completely the equations:

(i) \( \tan a \tan (\theta - a) + \tan \beta \tan (\theta - \beta) = \tan \frac{\beta - a}{2} (\tan a - \tan \beta) \),

and

(ii) \( \sin 3\theta = 4 \sin \theta \sin 2\theta \sin 4\theta \).

22. \( OA \) is a crank 2 feet long which rotates about \( O \); \( AB \) is a connecting rod to \( B \) which moves on a straight line passing through \( O \). Find the angles that the crank \( OA \) makes with \( OB \) when \( B \) has described respectively \( \frac{1}{2}, \frac{1}{4}, \) and \( \frac{1}{8} \) of its total travel from its extreme position, the length of \( AB \) being 5 feet.
23. From the top of a cliff, 200 feet high, two ships are observed at sea. The angle of depression of the one is 9° 10' and it is seen in a direction 30° North of East; the angle of depression of the other is 7° 30' and it is seen in a direction 25° South of East. What is the distance between the ships, and what is the bearing of the one as seen from the other?

24. Show that the radius of the circle, passing through the centre of the inscribed circle of a triangle and any two of the centres of the escribed circles, is equal to the diameter of the circumscribed circle of the triangle.

25. If
\[ \tan \left( \frac{\pi}{4} + \frac{y}{2} \right) = \tan^3 \left( \frac{\pi}{4} + \frac{x}{2} \right), \]
prove that
\[ \sin y = \sin x \frac{3 + \sin^2 x}{1 + 3 \sin^2 x}. \]

26. Prove that
\[ \sin \beta \sin \gamma \cos^2 \alpha \sin (\beta - \gamma) + \text{two similar expressions} = -\sin (\beta - \gamma) \sin (\gamma - \alpha) \sin (\alpha - \beta). \]

27. The legs of a pair of compasses are each 7 inches long, and the pencil leg has a joint at 4 inches from the common end of the two legs. The compasses are used to describe a circle of radius 4 inches, and the pencil leg is bent at the joint so that the pencil is perpendicular to the paper. Show that the angles of inclination of the two legs to the vertical are 19° 5' and 25° 20' approximately.

28. A tower stands in a field whose shape is that of an equilateral triangle and whose side is 80 feet. It subtends angles at the three corners whose tangents are respectively \(\sqrt{3}+1, \sqrt{2}, \sqrt{2}.\) Find its height.

29. Two circles of radii \(r_1\) and \(r_2\) cut at an angle \(\alpha\); show that the area common to them is
\[ (r_1^2 - r_2^2) \tan^{-1} \frac{r_2 \sin \alpha}{r_1 + r_2 \cos \alpha} + r_1^2 - r_2 \sin \alpha. \]

30. Find the simplest values of
\[ \tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}, \text{ and } \tan \left( \frac{1}{2} \sin^{-1} \frac{2x}{1+x^2} + \frac{1}{2} \cos^{-1} \frac{1-y^2}{1+y^2} \right). \]
31. Eliminate $a$ and $\beta$ from the equations
\[
\sin a + \sin \beta = l, \\
\cos a + \cos \beta = m,
\]
and
\[
\tan \frac{a}{2} \tan \frac{\beta}{2} = n.
\]

32. Find, by drawing graphs, how many real roots of the equation $x^2 \tan x = 1$ lie between 0 and $2\pi$.

33. Show that
\[
\cos 2a = 2 \sin^2 \beta + 4 \cos (a + \beta) \sin a \sin \beta + \cos 2(a + \beta).
\]

34. Prove that
\[
\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} = \frac{1}{8}.
\]

35. Solve the equations
\[
\sqrt{3} \sin 2A = \sin 2B, \\
\sqrt{3} \sin^2 A + \sin^2 B = \frac{1}{2} (\sqrt{3} - 1).
\]

36. If the tangents of the angles of a triangle are in arithmetic progression, prove that the squares of the sides are in the ratio
\[
x^2 (x^2 + 9) : (3 + x^2)^2 : 9 (1 + x^2),
\]
where $x$ is the least or greatest tangent.

37. $A$, $B$, $C$ are three points on a horizonal plane in the same straight line, $AB$ being 100 yards and $BC$ 150 yards. The angles of elevation of a balloon observed simultaneously from $A$, $B$, $C$ are $\alpha$, $\beta$, $\gamma$. Show that the height $h$ of the balloon in yards is given by
\[
h^2 (3 \cot^2 \alpha + 2 \cot^2 \gamma - 5 \cot^2 \beta) = 75,000.
\]

38. If $p$, $q$, $r$ are the perpendiculars from the vertices of a triangle upon any straight line meeting the sides externally in $D$, $E$, $F$, prove that
\[
a^2 (p - q) (p - r) + b^2 (q - r) (q - p) + c^2 (r - p) (r - q) = 4\Delta^2,
\]
where $\Delta$ is the area of the triangle.

Prove also that
\[
EF = \frac{2p\Delta}{(p - q) (p - r)}.
\]

39. The length of the side of a regular polygon of $n$ sides is $2l$, and the areas of the polygon and of the inscribed and circumscribed circles are $A$, $A_1$, and $A_2$; prove that
\[
A_2 - A_1 = \pi l^2 \text{ and } n^2 l^2 A_1 = \pi A^2.
\]

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40. Prove that in the triangle whose sides are 31, 56, and 64, one of the angles differs from a right angle by rather less than a minute of angle.

41. Show that
\[
\frac{1 + \sin A}{\cos A} + \frac{\cos B}{1 - \sin B} = \frac{2\sin A - 2\sin B}{\sin (A - B) + \cos A - \cos B}.
\]

42. Prove that
\[(1 + \sec 2\theta)(1 + \sec 4\theta)(1 + \sec 8\theta)\ldots(1 + \sec 2^n\theta) = \tan 2^n\theta \cdot \cot \theta.
\]

43. If the sides of a triangle are in arithmetic progression, and if its greatest angle exceeds the least angle by \(a\), show that the sides are in the ratio \(1 - x : 1 : 1 + x\), where \(x = \sqrt{\frac{1 - \cos a}{7 - \cos a}}\).

44. A tower stands on the edge of a circular lake \(ABCD\). The foot of the tower is at \(D\) and the angles of elevation of its top at \(A\), \(B\), \(C\) are respectively \(a\), \(\beta\), and \(\gamma\). If the angles \(BAC\), \(ACB\) are each \(\theta\), show that
\[2\cos \theta \cot \beta = \cot a + \cot \gamma.
\]

45. The internal bisectors of the angles of a triangle \(ABC\) meet the sides in \(D\), \(E\), and \(F\). Show that the area of the triangle \(DEF\) is equal to \(2\Delta abc/(b + c)(c + a)(a + b)\).

46. If \(\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi\), prove that
\[x^2 + y^2 + z^2 + 2xyz = 1.
\]

47. Eliminate \(\theta\) from the equations
\[\lambda \cos 2\theta = \cos (\theta + a),
\]
and
\[\lambda \sin 2\theta = 2\sin (\theta + a).
\]

48. A circle is described whose diameter is 6 inches; find an equation to determine the angle subtended at the centre by an arc which is such that the sum of the arc and its chord is 8 inches, and solve the equation by a graphic method.

49. Simplify
\[
\left[\frac{\cos (a + \beta)}{\cos (a - \beta)} - \frac{\cos (a + \gamma)}{\cos (a - \gamma)}\right]^2 + \left[\frac{\sin (a + \beta)}{\cos (a - \beta)} - \frac{\sin (a + \gamma)}{\cos (a - \gamma)}\right]^2.
\]

50. Show that
\[\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ = 1 + \sin^2 9^\circ + \sin^2 18^\circ.
\]
51. If $a, b, c$ are the sides of a triangle, $\lambda a, \lambda b, \lambda c$ the sides of a similar triangle inscribed in the former and $\theta$ the angle between the sides $a$ and $\lambda a$, prove that $2\lambda \cos \theta = 1$.

52. The top of a hill is observed from two stations $A$ and $B$ on the same level; $A$ is south of the hill and $B$ is north-east of $A$. If the angles of elevation from $A$ and $B$ are $9^\circ 30'$ and $7^\circ 30'$, find the compass bearing of $B$ from the hill.

53. The tangents at $B$ and $C$ to the circumcircle of a triangle $ABC$ meet in $A'$, and $O$ is the circumcentre. If the angle $OAA'$ is $\theta$, prove that

$$2 \tan \theta - \cot B \sim \cot C.$$ 

54. Find by a geometrical construction the number of values of $\cos \left( \frac{\pi}{5} \sin^{-1} a \right)$. Show that their product is $-\frac{1}{16} (1-a^2)$.

55. Show that the expression $\tan \left( \frac{x + a}{x - a} \right)$ cannot lie between the values

$$\tan^2 \left( \frac{\pi}{4} - a \right) \text{ and } \tan^2 \left( \frac{\pi}{4} + a \right).$$

56. Show that

$$\cos^4 \theta + \cos^4 \left( \theta + \frac{2\pi}{n} \right) + \cos^4 \left( \theta + \frac{4\pi}{n} \right) + \ldots \text{ to } n \text{ terms} = \frac{3n}{8}.$$

57. If

$$\{ \sin (a - \beta) + \cos (a + 2\beta) \sin \beta \}^2 = 4 \cos a \sin \beta \sin (a + \beta),$$

prove that

$$\tan a = \tan \beta \left\{ \frac{1}{(\sqrt{2} \cos \beta - 1)^2 - 1} \right\},$$

$a$ and $\beta$ being each less than a right angle.

58. Find all the values of $x$ which satisfy the equation

$$\tan (x + \beta) \tan (x + \gamma) + \tan (x + \gamma) \tan (x + a) + \tan (x + a) \tan (x + \beta) = 1.$$ 

59. $ABC$ is a triangle and $D$ is the foot of the perpendicular from $A$ upon $BC$. If $BC = 117$ feet, $\angle B = 43^\circ 14'$, and $\angle C = 61^\circ 27'$, find the length of $AD$.

60. The angles of elevation of the top of a mountain from three points $A, B, C$ in a base line are observed to be $\alpha, \beta, \gamma$ respectively. Prove that the height of the mountain is

$$(AB \cdot BC \cdot CA)^{\frac{3}{2}} \left( BC \cot^2 \alpha + CA \cot^2 \beta + AB \cot^2 \gamma \right)^{-\frac{1}{2}},$$

where regard is paid to the sense of the lines.
61. If the bisector of the angle $C$ of a triangle $ABC$ cuts $AB$ in $D$ and the circumcircle in $E$, prove that $CE : DE = (a + b)^2 : c^2$.

62. Eliminate $\theta$ from the equations

$$a \tan \theta + b \cot 2\theta = c,$$

and

$$a \cot \theta - b \tan 2\theta = c.$$  

63. Find, by a graph, an approximate value, correct to half a degree, of the equation $\cot x = \cos 2x$.

64. A man setting out a tennis court uses three strings of lengths 3 yds., 4 yds., and 4 yds. 2 ft. 10 ins. respectively to construct the right angle. Find the errors he makes in the angles of the court. "

65. If $n^2 \sin^2 (a + \beta) = \sin^2 a + \sin^2 \beta - 2 \sin a \sin \beta \cos (a - \beta)$,

show that

$$\tan a = \frac{1 + n}{1 - n} \tan \beta;$$

66. If the expression

$$\frac{A \cos (\theta + a) + B \sin (\theta + \beta)}{A' \sin (\theta + a) + B' \cos (\theta + \beta)}$$

retain the same value for all values of $\theta$, show that

$$AA' - BB' = (A'B - AB') \sin (a - \beta).$$

67. Show that the values of $\theta$ which are the roots of the equation

$$\sin 2\theta \cos^2 (a - \beta) - \sin 2a \cos^2 (\beta + \theta) - \sin 2\beta \sin^2 (a + \theta) = 0$$

are given by $(2n+1) \frac{\pi}{2} - \beta$ and $n\pi + a$, where $n$ is any positive or negative integer.

68. The three medians of a triangle $ABC$ make angles $\alpha$, $\beta$, $\gamma$ with each other. Prove that

$$\cot \alpha + \cot \beta + \cot \gamma + \cot A + \cot B + \cot C = 0.$$  

69. From each of two points, distant $2a$ apart, on one bank of a river the angular elevation of the top of a tower on the opposite bank is $a$, and from the point midway between these two points the angular elevation of the top of the tower is $\beta$. Find in terms of $a$, $\alpha$, $\beta$ the height of the tower and the breadth of the river.

If $a = 100$ yards, $\alpha = 22\frac{1}{2}^\circ$, and $\beta = 30^\circ$, show that the height of the tower is $150 \sqrt{2}$ feet.
70. If \( P, Q, R \) are the points of contact of the inscribed circle with the sides \( BC, CA, AB \) of a triangle, show that if the squares of \( PQ, QR, RP \) are in arithmetic progression, then the sides of the triangle are in harmonic progression.

71. Show that half the side of the equilateral triangle inscribed in a circle differs from the side of the regular inscribed heptagon by less than \( \frac{1}{60} \)th of the radius.

72. Show that the quantity
\[
\cos \theta \left( \sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right)
\]
always lies between the values \( \pm \sqrt{1 + \sin^2 \alpha} \).

73. Express \( 8 \sin \alpha \sin \beta \sin \gamma \sin \delta \) as a series of eight cosines.

74. If
\[
\sin^2 \phi = \frac{\cos 2\alpha \cos 2\beta}{\cos^2 (\alpha + \beta)},
\]
prove that
\[
\tan^2 \frac{\phi}{2} = \frac{\tan \left( \frac{\pi}{4} \pm \alpha \right)}{\tan \left( \frac{\pi}{4} \pm \beta \right)}.
\]

75. Given the base \( a \) of a triangle, the opposite angle \( A \), and the product \( k^2 \) of the other two sides, solve the triangle and show that there is no such triangle if \( a < 2k \sin \frac{A}{2} \).

76. At a point \( O \) on a horizontal plane the angles of elevation of two points \( P \) and \( Q \) on the side of a hill are found to be \( 38^\circ \) and \( 25^\circ \); the distance of \( A \), the foot of the hill, from \( O \) is 500 yards and the distance \( AQ \) is 320 yards, the whole figure being in a vertical plane. Prove that the distance \( PQ \) is 329 yards approximately, and find the slope of the hill.

77. \( I_1, I_2, \) and \( I_3 \) are the centres of the circles escribed to \( ABC \), and \( \rho_1, \rho_2, \rho_3 \) are the radii of the circles inscribed in the triangles \( BI_1C, CI_2A, AI_3B \). Show that
\[
\rho_1 : \rho_2 : \rho_3 : : \sin \frac{A}{2} : \sin \frac{B}{2} : \sin \frac{C}{2}.
\]

78. Two circles, the sum of whose radii is \( a \), are placed in the same plane with their centres at a distance \( 2a \), and an endless string is fully stretched so as partly to surround the circles and to cross between them. Show that the length of the string is \( \left( \frac{4\pi}{3} + 2\sqrt{3} \right) a \).
79. Show that
\[ 2 \tan^{-1} \frac{\sqrt{x^2 + a^2} - x + b}{\sqrt{x^2 - b^2}} + \tan^{-1} \frac{x \sqrt{a^2 - b^2}}{b \sqrt{x^2 + a^2 + a^2}} + \tan^{-1} \frac{\sqrt{a^2 - b^2}}{b} = n\pi. \]

80. Given that \(-2.45\) is an approximate value of \(x\) satisfying the equation \(3 \sin x = 2x + 3\), find a closer approximation. [Assume that \(2.45 \text{ radians} = 140^\circ 22' 30''\).]

81. Show that
\[ \sin A = \sin (36^\circ + A) - \sin (36^\circ - A) - \sin (72^\circ + A) + \sin (72^\circ - A). \]

82. Find the complete solution of the equations
\[ \tan 3\theta + \tan 3\phi = 2, \]
\[ \tan \theta + \tan \phi = 4. \]

83. If \(ABC\) be a triangle, and if
\[ \sin^3 \theta = \sin (A - \theta) \sin (B - \theta) \sin (C - \theta), \]
then
\[ \cot \theta = \cot A + \cot B + \cot C. \]

84. A ship steaming at a speed of 15 miles per hour towards a harbour \(A\) was observed from a station \(B\), 10 miles due west of \(A\), to lie \(42^\circ\) N. of E. If the ship reached the harbour after three-quarters of an hour, find its distance from \(B\) when first observed.

85. Show that the radius of the circle inscribed in the triangle formed by joining the centres of the escribed circles of a triangle \(ABC\) is
\[ 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \]
\[ \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}. \]

86. A polygon of \(n\) sides inscribed in a circle is such that its sides subtend angles \(2a, 4a, 6a...2na\) at the centre; prove that its area is to the area of the regular inscribed polygon of \(n\) sides in the ratio \(\sin na : n \sin a\).

87. Express the equation
\[ \cot^{-1} \left\{ \frac{y}{\sqrt{1 - x^2 - y^2}} \right\} = 2 \tan^{-1} \sqrt{\frac{3 - 4x^2}{4x^2}} - \tan^{-1} \frac{\sqrt{3 - 4x^2}}{x^2} \]
as a rational integral equation between \(x\) and \(y\).
88. If \( x_1, x_2, x_3, x_4 \) are the roots of the equation
\[
x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0,
\]
prove that
\[
\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta,
\]
where \( n \) is an integer.

89. If
\[
\frac{\sin (\theta - \beta) \cos a}{\sin (\phi - a) \cos \beta} + \frac{\cos (a + \theta) \sin 2\beta}{\cos (\phi - \beta) \sin 2\beta} = 0,
\]
and
\[
\frac{\tan \theta \tan a}{\tan \phi \tan \beta} + \frac{\cos (a - \beta)}{\cos (a + \beta)} = 0,
\]
show that
\[
\tan \theta = \frac{1}{2} (\tan \beta + \cot a) \quad \text{and} \quad \tan \phi = \frac{1}{2} (\tan a - \cot \beta).
\]

90. If \( \cos 3x = \frac{3}{8} \sqrt{6} \), show that the three values of \( \cos x \) are
\[
\frac{1}{2} \sqrt{6} \sin \frac{\pi}{10}, \quad \frac{1}{2} \sqrt{6} \sin \frac{\pi}{6} \quad \text{and} \quad -\frac{1}{2} \sqrt{6} \sin \frac{3\pi}{10}.
\]

91. The base \( a \) of a triangle and the ratio \( r (\lt 1) \) of the sides are given. Show that the altitude \( h \) of the triangle cannot exceed \( \frac{ar}{1 - r^2} \), and that when \( h \) has this value the vertical angle of the triangle is \( \frac{\pi}{2} - 2\tan^{-1} r \).

92. A railway curve, in the shape of a quadrant of a circle, has \( n \) telegraph posts at its ends and at equal distances along the curve. A man stationed at a point on one of the extreme radii produced sees the \( p \)th and \( q \)th posts from the end nearest him in a straight line. Show that the radius of the curve is
\[
\frac{a}{2} \cos (p + q) \phi \cosec p\phi \cosec q\phi, \quad \text{where} \quad \phi = \frac{\pi}{4(n - 1)}, \quad \text{and} \quad a \text{ is the distance from the man to the nearest end of the curve.}
\]

93. Show that the radii of the three escribed circles of a triangle are the roots of the equation
\[
x^3 - x^2 (1R + r) + xR^2 - Rr^2 = 0.
\]

94. Eliminate \( x \) and \( y \) from the equations
\[
\cos x + \cos y = a,
\]
\[
\cos 2x + \cos 2y = b,
\]
and
\[
\cos 3x + \cos 3y = c,
\]
giving the result in a rational form.
95. Sum the series
\[ \sin \theta \sin 2\theta \sin 3\theta + \sin 2\theta \sin 3\theta \sin 4\theta + \sin 3\theta \sin 4\theta \sin 5\theta + \ldots \text{ to } n \text{ terms.} \]

96. In a circle of radius 5 inches the area of a certain segment is 25 square inches. Find graphically the angle that is subtended at the centre by the arc of the segment.

97. Prove that
\[ 4 \sin 27^\circ = (5 + \sqrt{5})^{1/2} - (3 - \sqrt{5})^{1/2}. \]

98. If \( \cos (\beta - \gamma) + \cos (\gamma - a) + \cos (a - \beta) + 1 = 0, \)
show that \( \beta - \gamma, \gamma - a, \) or \( a - \beta \) is a multiple of \( \pi. \)

99. Given the product \( p \) of the sines of the angles of a triangle, and the product \( q \) of their cosines, show that the tangents of the angles are the roots of the equation
\[ qx^3 - px^2 + (1 + q)x - p = 0. \]

If \( p = \frac{1}{8} (3 + \sqrt{3}) \) and \( q = \frac{1}{8} (\sqrt{3} - 1), \) show that the angles of the triangle are \( 45^\circ, 60^\circ \) and \( 75^\circ. \)

100. Observations on the position of a ship are made from a fixed station. At one instant the bearing of the ship is \( a_1 \) West of North. Ten minutes later the ship is due North and after a further interval of ten minutes its bearing is \( a_2 \) East of North. Assuming that the speed and direction of motion of the ship have not changed, show that its course is \( \theta \) East of North where
\[
\tan \theta = \frac{2 \sin a_1 \sin a_2}{\sin (a_1 - a_2)}.
\]

101. A hill on a level plane has the form of a portion of a sphere. At the bottom the surface slopes at an angle \( a \) and from a point on the plain distant \( a \) from the foot of the hill the elevation of the highest visible point is \( \beta. \) Prove that the height of the hill above the plain is
\[
\frac{a \sin \beta \sin^2 \frac{a}{2}}{\sin^2 \frac{a - \beta}{2}}.
\]

102. If \( D, E, F \) be the feet of the perpendiculars from \( ABC \) on the opposite sides and \( r, r_1, r_2, r_3 \) be the radii of the circles inscribed in the triangles \( DEF, AEF, BFD, CDE, \) prove that \( r^3 = 2Rr_1r_2r_3. \)
103. \(O\) is the centre of a circular field and \(A\) any point on its boundary; a horse, tethered by a rope fastened at one end at \(A\), can graze over \(\frac{1}{n}\)th of the field; if \(B\) be the furthest point of the boundary that he can reach and \(\angle AOB = \theta\), prove that
\[
\sin \theta + (\pi - \theta) \cos \theta = \left(1 - \frac{1}{n}\right) \pi.
\]

104. Solve the equation
\[
\theta = \tan^{-1} \left(2 \tan^2 \theta\right) - \frac{1}{4} \sin^{-1} \frac{3 \sin 2\theta}{5 + 4 \cos 2\theta}.
\]

105. If \(\cos^2 \theta = \frac{m^2 - 1}{3}\) and \(\tan^3 \frac{\theta}{2} = \tan a\), prove that
\[
\cos^2 a + \sin^2 a = \left(\frac{2}{m}\right)^3.
\]

106. A man walks on a horizontal plane a distance \(a\), and then through a distance \(a\) at an angle \(a\) with his previous direction. After he has done this \(n\) times, the change of his direction being always in the same sense, show that he is distant
\[
\frac{a \sin \frac{na}{2}}{\sin \frac{a}{2}}
\]
from his starting point, and that this distance makes an angle \((n - 1) \frac{a}{2}\) with his original direction.

107. Prove that
\[
\frac{\tan (\gamma - \delta)}{\tan (\delta - \beta)} + \frac{\tan (\beta - \gamma)}{\tan (\gamma - \delta)} + \frac{\tan (\beta - \gamma)}{\tan (\gamma - \delta)} + \frac{\tan (\beta - \gamma)}{\tan (\gamma - \delta)} = 0.
\]

108. A meteor moving in a straight line passes vertically above two points, \(A\) and \(B\), in a horizontal plane, 1000 feet apart. When above \(A\) it has altitude 50° as seen from \(B\), and when above \(B\) it has altitude 40° as seen from \(A\). Find the distance from \(A\) at which it will strike the plane, correct to the nearest foot.

109. The face of a hill is a plane inclined at an angle \(\theta\) to the horizontal. From two points at the foot of the hill two men walk up it along straight paths lying in vertical planes perpendicular to one another. If they meet after having walked distances \(a\) and \(b\) respectively, show that they are then at a vertical height \(h\) given by the smaller root of the quadratic
\[
(2 - \sin^2 \theta) h^4 - (a^2 + b^2) h^2 + a^2 b^2 \sin^2 \theta = 0.
\]
110. Show that, if \( a, \beta, \gamma, \delta \) are roots of

\[
\tan \left( \theta + \frac{\pi}{4} \right) = 3 \tan 3\theta,
\]

no two of which have equal tangents, then

\[
\tan a + \tan \beta + \tan \gamma + \tan \delta = 0.
\]

111. If \( \theta_1, \theta_2, \theta_3, \theta_4 \) be roots of the equation

\[
\sin (\theta + a) = k \sin 2\theta,
\]

no two of which differ by a multiple of \( 2\pi \), prove that

\[
\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1) \pi.
\]

112. Prove, by means of projections, the theorems of Art. 213.

113. Prove the identities

(i) \[
\sin a + \sin \beta + \sin \gamma = \sin (a + \beta + \gamma)
\]

\[
= 4 \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + a}{2} \sin \frac{a + \beta}{2};
\]

(ii) \[
\cos^2 a \sin 2(\beta - \gamma) + \cos^2 \beta \sin 2(\gamma - a) + \cos^2 \gamma \sin 2(a - \beta)
\]

\[
+ 2 \sin (\beta - \gamma) \sin (\gamma - a) \sin (a - \beta) = 0.
\]

114. Show that the equation

\[
\sec \theta + \csc \theta = c
\]

has two roots between 0 and \( 2\pi \), if \( c^2 < 8 \), and four roots if \( c^2 > 8 \).

115. If the external bisectors of the angles of the triangle \( A_0B_0C_0 \) form a triangle \( A_1B_1C_1 \), and if the external bisectors of the triangle \( A_1B_1C_1 \) form a triangle \( A_2B_2C_2 \), and so on, show that the angle \( A_n \)

of the \( n \)th derived triangle is \( \frac{\pi}{3} + \left( \frac{-1}{2} \right)^n \left( A - \frac{\pi}{3} \right) \), and that the triangles tend to become equilateral.

116. From a certain station \( A \) the angular elevation of a mountain peak \( P \), to the North of \( A \), is \( a \). A hill, of height \( h \) above \( A \), is ascended. From \( B \), the top of this hill, the angular elevation of \( P \) is \( \beta \), the bearing of \( A \) is \( \delta \) West of South, and the bearing of \( P \) is \( \gamma \) North of \( A \). Show that the height of \( P \) above \( A \) is

\[
\frac{h \tan a \sin \gamma}{\tan a \sin \gamma - \tan \beta \sin \delta}.
\]

117. A man at the bottom of a hill observes an object, half a mile distant, at the same level as himself. He then walks 200 yards up the hill and observes that the angle of depression of the object is \( 2^\circ 30' \) and that the direction to it makes an angle of \( 75^\circ \) with the direction to his starting point. Find to the nearest minute the angle which his path makes with the horizontal.
118. If $2\phi_1$, $2\phi_2$, $2\phi_3$ are the angles subtended by the circle escribed to the side $a$ of a triangle at the centres of the inscribed circle and the other two inscribed circles, prove that

$$\sin \phi_1 \sin \phi_2 \sin \phi_3 = \frac{r_1^2}{16R^2}.$$ 

119. A regular polygon of $n$ sides is placed with one side in contact with a fixed straight line, and is turned about one extremity of this side until the next side is in contact with the straight line and so on for a complete revolution; show that the length of the path described by any one of the angular points of the polygon is $\frac{4\pi R}{n} \cot \frac{\pi}{2n}$, where $R$ is the radius of the circle circumscribing the polygon.

Show also that the sum of the areas of the sectors of the circles described by the angular point is $2\pi R^2$.

120. Eliminate $\theta$ and $\phi$ from the equations

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1,$$

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,$$

and

$$\frac{\cos \theta \cos \phi}{a^2} + \frac{\sin \theta \sin \phi}{b^2} = 0.$$

121. Prove the assumption made in Art. 227, by taking a very large number of points $Q_1$, $Q_2$, $Q_3$... between $P$ and $P'$ and producing $PQ_1$, $Q_1Q_2$, $Q_2Q_3$... to meet $TP'$ in $R_1$, $R_2$, $R_3$... and using the proposition that two sides of a triangle are always greater than the third.

If $AP_1P_2...P_nB$, $AQ_1Q_2...Q_nB$ be any two convex broken lines ending in the same points $AB$, of which the first is wholly outside the second, show that the first is greater than the second.

122. Prove that

$$\frac{\sin(\theta - \gamma - \alpha)}{\sin(a - \alpha \beta)} \frac{\sin(\beta - \alpha)}{\sin(\gamma - a)} + \frac{\sin(\theta - \alpha - \beta)}{\sin(\gamma - \beta)} \frac{\sin(\alpha - \beta)}{\sin(\gamma - a)} + \frac{\sin(\theta - \beta - \gamma)}{\sin(\alpha - \gamma)} \frac{\sin(\theta - \gamma - a)}{\sin(\beta - \gamma)} = 1.$$ 

123. If

$$(\sin^2 \phi - \sin^2 \psi) \cot \theta + (\sin^2 \psi - \sin^2 \theta) \cot \phi + (\sin^2 \theta - \sin^2 \phi) \cot \psi = 0,$$

then either the difference of two angles or the sum of all three is a multiple of $\pi$. 
124. A hill, standing on a horizontal plane, has a circular base and forms part of a sphere. At two points on the plane, distant \( a \) and \( b \) from the base, the angular elevations of the highest visible points on the hill are \( \theta \) and \( \phi \). Prove that the height of the hill is

\[
2 \left[ \frac{\left( b \cot \frac{\phi}{2} \right)^\frac{1}{2} - \left( a \cot \frac{\theta}{2} \right)^\frac{1}{2}}{\cot \frac{\phi}{2} - \cot \frac{\theta}{2}} \right]^2.
\]

125. There is a hemispherical dome on the top of a tower; on the top of the dome stands a cross; at a certain point the elevation of the cross is observed to be \( \alpha \), and that of the dome to be \( \beta \); at a distance \( a \) nearer the dome, the cross is seen just above the dome, when its elevation is observed to be \( \gamma \); prove that the height of the centre of the dome above the ground is

\[
\frac{a \sin \gamma}{\sin (\gamma - a)} \cdot \frac{\cos \alpha \sin \beta - \sin \alpha \cos \gamma}{\cos \beta - \cos \gamma}.
\]

126. If \( \sin^2 A + \sin^2 B + \sin^2 C = 1 \), show that the circumscribed circle of the triangle \( ABC \) cuts its nine-point circle orthogonally.

127. A point \( O \) is situated on a circle of radius \( R \), and with centre \( O \) another circle of radius \( \frac{3R}{2} \) is described. Inside the crescent-shaped area intercepted between these circles a circle of radius \( \frac{R}{3} \) is placed. Show that if the small circle moves in contact with the original circle of radius \( R \), the length of arc described by its centre in moving from one extreme position to the other is \( \frac{1}{2} \pi R \).

128. Eliminate \( x \) and \( y \) from the equations

\[
\sin x + \sin y = a,
\]
\[
\cos x + \cos y = b,
\]
and \( \tan x + \tan y = c \).

129. If \( 2 \cos n\theta \) be denoted by \( u_n \), show that

\[
u_{n+1} = u_{n+1}u_n - u_{n-1}.
\]

Hence show that

\[
2 \cos 7\theta = u_1^7 - 7u_1^5 + 14u_1^3 - 7u_1.
\]
130. Show by a graph that \( \cdot 74 \) is an approximate solution of the equation \( \cos x = x \) (where \( x \) is measured in radians), and prove that this is the only real root.

Further, by putting \( x = \cdot 74 + y \), where \( y \) is small, prove that a still nearer value of \( x \) is \( \cdot 7391 \) so that the angle required is \( 42^\circ \ 21' \) to the nearest minute.

131. Show that
\[
\frac{\sin (x - \beta) \sin (x - \gamma)}{\sin (\alpha - \beta) \sin (\alpha - \gamma)} \sin 2 (x - \alpha) + \text{two similar terms} = 0.
\]

132. If \( ABC \) is a triangle, prove that
\[
\sin^3 A \cos (B - C) + \sin^3 B \cos (C - A) + \sin^3 C \cos (A - B) = 3 \sin A \sin B \sin C.
\]

133. A man notices two objects in a straight line due west. After walking a distance \( c \) due north he observes that the objects subtend an angle \( \alpha \) at his eye; and, after walking a further distance \( c \) due north, an angle \( \beta \). Show that the distance between the objects is
\[
\frac{3c}{2 \cot \beta - \cot \alpha}.
\]

134. The side of a hill is plane and inclined at an angle \( \alpha \) to the horizon; a road on it is in a vertical plane making an angle \( \beta \) with the vertical plane through the line of greatest slope; prove that the inclination of the road to the horizontal is \( \tan^{-1} (\tan \alpha \cos \beta) \).

135. Show that the line joining the incentre to the circumcentre of a triangle \( ABC \) is inclined to \( BC \) at an angle
\[
\tan^{-1} \left( \frac{\cos B + \cos C - 1}{\sin B \sin C} \right).
\]

136. Eliminate \( \theta \) from the equations
\[
x \sin \theta - y \cos \theta = -\sin 4\theta,
\]
and
\[
x \cos \theta + y \sin \theta = \frac{1}{2} - \frac{3}{2} \cos 4\theta.
\]

137. A regular polygon is inscribed in a circle; show that the arithmetic mean of the squares of the distances of its corners from any point (not necessarily in its plane) is equal to the arithmetic mean of the sum of the squares of the longest and shortest distances of the point from the circle.
138. Three points $A$, $B$, $C$ lie in a straight line and $AB$ is to $BC$ as $m$ to $n$. Through $A$, $B$, $C$ are drawn parallel straight lines $AX$, $BY$, $CZ$. A point $P$ moves on $AX$ and a point $R$ on $CZ$ so that at any time $t$ the distance $AP$ is equal to $a_1 + a_2 \sin (nt + a)$ and the distance $CR$ is equal to $c_1 + c_2 \sin (nt + \gamma)$, and the straight line $PR$ cuts $BY$ in $Q$. Express the distance $BQ$ in a similar form.

139. Prove that
\[
\sin (\beta - \gamma) \sin 3a + \sin (\gamma - a) \sin 3\beta + \sin (\alpha - \beta) \sin 3\gamma = 4 \sin (\beta - \gamma) \sin (\gamma - a) \sin (\alpha - \beta) \sin (\alpha + \beta + \gamma).
\]

140. Prove that
\[
\sin (\beta - \gamma) \cos 3a + \sin (\gamma - a) \cos 3\beta + \sin (\alpha - \beta) \cos 3\gamma = 4 \sin (\beta - \gamma) \sin (\gamma - a) \sin (\alpha - \beta) \cos (\alpha + \beta + \gamma).
\]

141. If
\[
\sin (x + 3a) \sin (\beta - \gamma) + \sin (x + 3\beta) \sin (\gamma - a) + \sin (x + 3\gamma) \sin (\alpha - \beta) = 4 \sin (\beta - \gamma) \sin (\gamma - a) \sin (\alpha - \beta),
\]
prove that
\[
x + a + \beta + \gamma = (2n + \frac{1}{2}) \pi.
\]

142. If $A + B + C = 2\pi$, show that
\[
\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} - \sin \frac{3A}{2} \sin \frac{3B}{2} \sin \frac{3C}{2}.
\]

143. A man walks in a horizontal circle round the foot of a flagstaff, which is inclined to the vertical, the foot of the flagstaff being the centre of the circle. The greatest and least angles which the flagstaff subtends at his eye are $\alpha$ and $\beta$; and when he is midway between the corresponding positions the angle is $\theta$. If the man’s height be neglected, prove that
\[
\tan \theta = \sqrt{\sin^2 \frac{\alpha - \beta}{2} + 4 \cos^2 \alpha \cos^2 \beta} / \sin (\alpha + \beta).
\]

144. Two lines, inclined at an angle $\gamma$, are drawn on an inclined plane and their inclinations to the horizon are found to be $\alpha$ and $\beta$ respectively; show that the inclination of the plane to the horizon is
\[
\sin^{-1} \left\{ \cosec \gamma \sqrt{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \gamma} \right\},
\]
and that the angle between one of the given pair of lines and the line of greatest slope on the inclined plane is
\[
\tan^{-1} \left\{ \frac{\sin \beta - \sin \alpha \cos \gamma}{\sin \alpha \sin \gamma} \right\}.
\]
145. Show that the line joining the orthocentre to the circumcentre of a triangle $ABC$ is inclined to $BC$ at an angle
\[ \tan^{-1}\left(\frac{3 - \tan B \tan C}{\tan B - \tan C}\right) . \]

146. Eliminate $\theta$ from the equations
\[ \frac{\cos(\alpha - 3\theta)}{\cos^3 \theta} = \frac{\sin(\alpha - 3\theta)}{\sin^3 \theta} = m. \]

147. $A_1 A_2 A_3 \ldots A_n$ is a regular polygon of $n$ sides circumscribed to a circle of centre $O$ and radius $a$. $P$ is any point distant $c$ from $O$. Show that the sum of the squares of the perpendiculars from $P$ on the sides of the polygon is $n \left( a^2 + \frac{c^2}{2} \right)$.

148. $AB$ is an arc of a circle which subtends an angle of $2\theta$ at its centre, and the tangents at $A$ and $B$ meet in $T$. By graphic methods find the value of $\theta$ to the nearest degree,

(i) when the area between $TA$, $TB$ and the arc $AB$ is equal to the area of the circle;

(ii) when the sum of the lengths of $TA$ and $TB$ is equal to the sum of the lengths of the arc $AB$ and the chord $AB$.

149. Show that the angles of a triangle satisfy the relations

(i) \[ \sin^3 A + \sin^3 B + \sin^3 C = 3 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2} ; \]

(ii) \[ \sin^4 A + \sin^4 B + \sin^4 C = \frac{3}{8} + 2 \cos A \cos B \cos C + \frac{1}{4} \cos 2A \cos 2B \cos 2C. \]

150. From a point $O$ a man is observed to be walking in a straight path up a hill, and from two sets of observations on his apparent size, made as he passes two points $P$, $Q$, it is found that $OP/OQ = \lambda$, and the angle $POQ = \gamma$. The elevations of $P$ and $Q$ above $O$ being $\alpha$ and $\beta$, prove that the inclination $\phi$ of the path to the horizon is given by
\[ \sin^2 \phi = (\alpha \sin \alpha - \sin \beta)^2 / (\lambda^2 - 2\lambda \cos \gamma + 1). \]

151. Eliminate $\theta$ from the equations
\[ x + a = a (2 \cos \theta - \cos 2\theta), \]
and
\[ y = a (2 \sin \theta - \sin 2\theta). \]
152. If from any point in the plane of a regular polygon perpendiculars are drawn on the sides, show that the sum of the squares of these perpendiculars is equal to the sum of the squares on the lines joining the feet of the perpendiculars with the centre of the polygon.

153. A horse is tied to a peg in the centre of a rectangular field of sides $a$ and $2a$; if he can graze over just half of the field, show that the length of the rope by which he is tethered is approximately $0.583a$.

154. Show that the equation

$$\sin(\theta + \lambda) = a \sin 2\theta + b$$

has four roots whose sum is an odd multiple of two right angles.

155. If $\theta$ is a positive acute angle, show that $\frac{\theta}{\sin \theta}$ continually increases, and $\frac{\theta}{\tan \theta}$ continually decreases, as $\theta$ increases.

156. If $\sin x = m \sin y$ where $m$ is greater than unity, show that as $x$ increases from zero to a right angle $\frac{\tan x}{\tan y}$ continually increases, and that its values, when $x$ is zero and a right angle, are $m$ and $\infty$ respectively.

157. Prove that

$$\sin^3(\beta - \gamma) \sin^3(\alpha - \delta) + \sin^3(\gamma - \alpha) \sin^3(\beta - \delta) + \sin^3(\alpha - \beta) \sin^3(\gamma - \delta)$$

$$= 3 \sin(\alpha - \beta) \sin(\beta - \gamma) \sin(\gamma - \alpha) \sin(\alpha - \delta) \sin(\beta - \delta) \sin(\gamma - \delta).$$

158. Prove that

$$\Sigma \cos(3a - \beta - \gamma - \delta)$$

$$= 4 \cos(a + \beta - \gamma - \delta) \cos(a + \gamma - \beta - \delta) \cos(a + \delta - \beta - \gamma).$$

159. Show that

$$\sin(\alpha + \beta + \gamma) \cos a \sin \beta \sin \gamma + \cos(\alpha + \beta + \gamma) \sin a \sin \beta \sin \gamma$$

$$- \sin(\alpha + \beta + \gamma) \cos a \cos \beta \cos \gamma - \cos(\alpha + \beta + \gamma) \sin a \cos \beta \cos \gamma$$

$$+ \sin(\alpha + \beta) \cos(\beta + \gamma) \cos(\gamma + a) + \cos(\alpha + \beta) \cos(\beta + \gamma) \sin(\gamma + a) = 0.$$ 

160. Show that

$$\sin^2 a \sin(\beta - \gamma) \sin(\gamma - \delta) \sin(\delta - \beta)$$

$$- \sin^2 \beta \sin(\gamma - \delta) \sin(\delta - a) \sin(\alpha - \gamma)$$

$$+ \sin^2 \gamma \sin(\delta - a) \sin(\alpha - \beta) \sin(\beta - \delta)$$

$$- \sin^2 \delta \sin(\alpha - \beta) \sin(\beta - \gamma) \sin(\gamma - a) = 0.$$
161. Simplify the expression \( PQ - RS \), where
\[
P = x \cos (a + \beta) + y \sin (a + \beta) - \cos (a - \beta),
\]
\[
Q = x \cos (\gamma + \delta) + y \sin (\gamma + \delta) - \cos (\gamma - \delta),
\]
\[
R = x \cos (a + \gamma) + y \sin (a + \gamma) - \cos (a - \gamma),
\]
and
\[
S = x \cos (\beta + \delta) + y \sin (\beta + \delta) - \cos (\beta - \delta).
\]

162. If
\[
a^2 + b^2 - 2ab \cos \alpha = c^2 + d^2 - 2cd \cos \gamma,
\]
\[
b^2 + c^2 - 2bc \cos \beta = a^2 + d^2 - 2ad \cos \delta,
\]
and
\[
ab \sin \alpha + cd \sin \gamma = bc \sin \beta + ad \sin \delta,
\]
show that
\[
\cos (a + \gamma) = \cos (\beta + \delta).
\]

163. Show that the solution of the equation
\[
\begin{vmatrix}
1, & \cos \theta, & 0, & 0 \\
\cos \theta, & 1, & \cos \alpha, & \cos \beta \\
0, & \cos \alpha, & 1, & \cos \gamma \\
0, & \cos \beta, & \cos \gamma, & 1 \\
\end{vmatrix}
= 0
\]
is \( \theta = n\pi + (-1)^n \sin^{-1} \left\{ \frac{\sqrt{\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma}}{\sin \gamma} \right\} \).

164. In any triangle \( ABC \), show that
\[
\cos mA + \cos mB + \cos mC - 1 = \pm 4 \sin \frac{mA}{2} \sin \frac{mB}{2} \sin \frac{mC}{2},
\]
according as \( m \) is of the form
\[
4n + 1 \text{ or } 4n + 3.
\]

165. Show that, in any triangle \( ABC \),
(i) \( a^2 \cos B \cos C + b^2 \cos C \cos A + c^3 \cos A \cos B = abc \left( 1 - 2 \cos A \cos B \cos C \right) \),

and
(ii) \( \sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1} \cdot 4 \sin mA \sin mB \sin mC \).

166. If \( A, B, C \) are the angles of a triangle, prove that
\[
\tan^{-1} (\cot B \cot C) + \tan^{-1} (\cot C \cot A) + \tan^{-1} (\cot A \cot B) = \tan^{-1} \left\{ 1 + \frac{8 \cos A \cos B \cos C}{\sin^2 2A + \sin^2 2B + \sin^2 2C} \right\}.
\]

167. Through the angular points \( A, B, C \) of a triangle straight lines are drawn making the same angle \( a \) with \( AB, BC, CA \) respectively; show that the sides of the triangle thus formed bear to the sides of the triangle \( ABC \) the ratio
\[
\cos a - \sin a (\cot A + \cot B + \cot C) : 1.
\]
168. A cylindrical tower is surmounted by a cone; from a point on the ground the angles of elevation of the nearest point of the top of the tower and of the top of the cone are $\alpha$ and $\beta$, and from a point nearer to the tower by a distance $a$ these angles are $\gamma$ and $\delta$. Show that the heights above the ground of the tops of the tower and cone are

\[ a \sin \alpha \sin \gamma \csc (\gamma - a) \quad \text{and} \quad a \sin \beta \sin \delta \csc (\delta - \beta), \]

and that the diameter of the tower is

\[ 2a \sin \beta \cos \delta \csc (\delta - \beta) - 2a \sin \alpha \cos \gamma \csc (\gamma - a). \]

169. In order to find the dip of a stratum of rock below the surface of the ground, vertical holes are bored at three points of a horizontal square; the depths of the stratum at these points are found to be $a$, $b$, and $c$. Show that the dip of the stratum is

\[ \tan^{-1} \frac{\sqrt{(a-b)^2+(b-c)^2}}{d}, \]

where $d$ is the side of the square.

170. A tunnel is to be bored from $A$ to $B$ which are two places on the opposite sides of a mountain. From $A$ and $B$ the elevations of a distant point $C$ are found to be $a$ and $\beta$, and the angle $ACB$ is found to be $\gamma$; also the lengths $AC$, $BC$ are known to be $a$ and $b$. Show that the height ($h$) of $B$ above $A$ is $a \sin \alpha - b \sin \beta$, that the length ($k$) of $AB$ is $\sqrt{a^2+b^2-2ab \cos \gamma}$, and that $AB$ is inclined at $\sin^{-1} \frac{h}{k}$ to the horizontal and at $\sin^{-1} \frac{b \sin \gamma}{k}$ to the line $AC$.

171. A man walks up a hill of elevation $\phi$ in a direction making an angle $\lambda$ with the line of greatest slope; when he has walked up a distance $m$ he observes that $\alpha$ is the angle of depression of an object situated in the horizontal plane through the foot of the hill and in the vertical plane through the path he is taking; after walking a further distance $n$, he observes that the angle of depression of the same object is $\beta$. Show that the elevation $\phi$ is given by the equation

\[ \left\{ \frac{m}{n} (\cot \beta - \cot \alpha) + \cot \beta \right\}^2 + 1 = \csc^2 \phi \sec^2 \lambda. \]

172. $A$, $B$, $C$ are three mountain peaks of which $A$ is the lowest and $B$ is at a known height $h$ above $A$. At $A$ the elevations of $B$ and $C$ are found to be $\beta$ and $\gamma$, and the angle between the vertical planes through $AB$, $AC$ is found to be $\theta$. At $B$ the angle between the vertical planes through $BA$ and $BC$ is found to be $\phi$. Show that the height of $C$ above $A$ is $h \cot \beta \tan \gamma \sin \phi \csc (\theta + \phi)$. 
173. Two straight paths $BC$, $CA$ on a plane hill-side have lengths $a$, $b$ respectively and have the same upward gradient of 1 in $m$ (1 vertical in $m$ horizontal) while the gradient from $B$ to $A$ is 1 in $p$. Show that the inclination of the plane of the hill to the horizontal is $a$ where

$$4ab \cot^2 a = (a + b)^2 p^2 - (a - b)^2 m^2.$$  

174. Show that the distance between the centres of the inscribed and nine-point circles is equal to $R \frac{R}{2} - r$. Hence deduce Feuerbach's Theorem, that the in-circle and nine-point circles of any triangle touch one another.

175. $ABCD$ is a quadrilateral such that $AB = 3$, $BC = 4$, $CD = 5$ and $DA = 6$ feet, and its area is $3\sqrt{3} + 9$ square feet. Show that there are two quadrilaterals satisfying these conditions, for which the values of the angle $B$ are respectively $60^\circ$ and $\cos^{-1} \left[ -1 - 42\sqrt{3} \right]$, i.e. $60^\circ$ and $175^\circ 15'$ nearly.

176. Eliminate $a$, $b$, $c$ from the equations

$$a \cos \alpha + b \cos \beta + c \cos \gamma = 0,$$

$$a \sin \alpha + b \sin \beta + c \sin \gamma = 0,$$

and

$$a \sec \alpha + b \sec \beta + c \sec \gamma = 0.$$

177. Eliminate $\theta$ from the equations

$$\tan (\theta - a) + \tan (\theta - \beta) = x,$$

and

$$\cot (\theta - a) + \cot (\theta - \beta) = y.$$  

178. Eliminate $\phi$ from the equations

$$x \cos 3\phi + y \sin 3\phi = b \cos \phi,$$

and

$$x \sin 3\phi + y \cos 3\phi = b \cos \left( \phi + \frac{\pi}{6} \right).$$  

179. Two regular polygons, of $m$ and $n$ sides, are inscribed in the same circle, of radius $a$; show that the sum of the squares of all the chords which can be drawn to join a corner of one polygon to a corner of the other is $2mna^2$.

180. There are $n$ stones arranged at equal intervals round the circumference of a circle; compare the labour of carrying them all to the centre with that of heaping them all round one of the stones; and prove that, when the number of stones is indefinitely increased, the ratio is that of $\pi : 4$.  

21—2
181. By drawing a graph, or otherwise, find the number of roots of the equation
\[ x + 2 \tan x = \frac{\pi}{2} \]
lying between 0 and \(2\pi\), and find the approximate value of the largest of these roots.
Verify your result from the Tables.

182. Find the least positive value of \(x\) satisfying the equation
\[ \tan x - x = \frac{1}{2}. \]

183. Draw the graph of the function \(\sin^2 x\) and show from it that, if \(a\) is small and positive, the equation
\[ x - a = \frac{\pi}{2} \sin^2 x \]
has three real roots.

184. Show that approximations to the larger real roots of the equation
\[ ax + b = \tan \frac{\pi c x}{2} \]
are given by
\[ x = \frac{m}{c} - \frac{2}{\pi (am + bc)}, \]
where \(m\) is any large odd integer.

185. By a graph determine approximately the numerically smallest positive and negative roots of the equation
\[ x^2 \sin \pi x = 1. \]
Prove that the large roots of this equation are given approximately by
\[ x = n + \frac{(-1)^n}{n^2 \pi}, \]
where \(n\) is large.

186. Show that the root of the equation
\[ \tan x = 2x, \]
which lies between 0 and \(\frac{\pi}{2}\), is equal approximately to 1.1654, given that \(\tan 1.1519 = 2.2460\) and \(\tan 1.1694 = 2.3559\).

187. Show that \(\tan \theta\) is always greater than
\[ \theta + \frac{\theta^3}{3} + \frac{\theta^5}{15} + \cdots + \frac{\theta^{2n+1}}{4^n - 1} + \cdots, \]
if \(\theta\) be an acute angle.
188. Show that the equation
\[ \cos (2\theta - a) + a \cos (\theta - \beta) + b = 0, \]
where \( a, b, \alpha, \beta \) are constants, has four sets of roots; and denoting any
four roots of different sets by \( \theta_1, \theta_2, \theta_3, \theta_4 \), prove that
\[ \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2a \]
is an even multiple of \( \pi \).

189. The equation
\[ \cot (\theta + a) + \cot (\theta + \beta) + \cot (\theta + \gamma) \]
\[ = \csc (\theta + a) + \csc (\theta + \beta) + \csc (\theta + \gamma) \]
is satisfied by values of \( \theta \) equal to \( \theta_1, \theta_2, \) and \( \theta_3 \), no two of which differ
by a multiple of four right angles. Show that
\[ \theta_1 + \theta_2 + \theta_3 + \alpha + \beta + \gamma \]
is equal to a multiple of \( 2\pi \).

190. Show that, in general, the equation
\[ A \sin^3 x + B \cos^3 x + C = 0 \]
has six distinct roots, no two of which differ by \( 2\pi \), and that the tangent
of their semi-sum is \( -\frac{A}{B} \).

191. Show that the equation
\[ \tan (\theta - a) + \sec (\theta - \beta) = \cot \gamma \]
has four roots (not differing by multiples of \( 2\pi \)) which satisfy the relation
\[ \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2(n\pi + \alpha + \beta - \gamma). \]

192. Show that if \( a, \beta, \gamma \) are three values of \( x \) satisfying the equation
\[ \sin 2\theta (a \sin x + b \cos x) = \sin 2\alpha (a \sin \theta + b \cos \theta) \]
and not differing from one another, or from \( \theta \) by a multiple of \( 2\pi \), then
\[ \tan \frac{a}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\theta}{2} + 1 = 0. \]

193. Prove that if \( \theta_1, \theta_2, \theta_3, \theta_4 \) be four distinct roots of the equation
\[ a \cos 2\theta + b \sin 2\theta + c \cos \theta + d = 0, \]
then
\[ \sum \sin \frac{\theta_2 + \theta_3 + \theta_4 - \theta_1}{2} = 0. \]
194. Prove the relation

\[ \cos^{-1} x_0 = \frac{\sqrt{1 - x_0^2}}{x_1 \cdot x_2 \cdot x_3 \ldots \text{ad inf.}} \]

where the successive quantities \( x_r \) are connected by the relation

\[ x_{r+1} = \sqrt{\frac{1}{2} \left(1 + x_r\right)} \]

195. If \( a, b \) are positive quantities and if

\[ a_1 = \frac{a + b}{2}, \quad b_1 = \sqrt{a_1 b}, \quad a_2 = \frac{a_1 + b_1}{2}, \quad b_2 = \sqrt{a_2 b_1} \]

and so on, show that

\[ a_\infty = b_\infty = \frac{\sqrt{b^2 - a^2}}{\cos^{-1} \frac{a}{b}} \]

Hence show that the value of \( \pi \) may be found.

196. If the equation

\[ a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0 \]

holds for all values of \( x \), where all the constants \( a_1, a_2, \ldots \) are independent of \( x \), then each of these constants must be zero.
ANSWERS.

I. (Page 5.)

1. \( \frac{2}{3} \).
2. \( \frac{301}{360} \).
3. \( \frac{45569}{64800} \).
4. 1\( \frac{5}{6} ^\circ \).
5. 2\( \frac{3}{8} \)\( \frac{61}{1080} \)\( ^\circ \).
6. 4\( \frac{3}{8} \)\( \frac{8}{83} \)\( ^\circ \).
7. 33\( \frac{3}{3} \)\( \frac{3}{3} \)\( ^\circ \).
8. 90\( ^\circ \).
9. 153\( \frac{8}{8} \)\( \frac{88}{88} \)\( ^\circ \).
10. 39\( \frac{7}{6} \)\( \frac{38}{38} \)\( ^\circ \).
11. 261 34\( \frac{4}{4} \)\( ^\circ \).
12. 528\( \frac{3}{3} \)\( \frac{3}{3} \)\( ^\circ \).
13. 1\( \frac{1}{3} \) rt. \( \angle \); 108\( ^\circ \).
14. \( \cdot 453524 \) rt. \( \angle \); 40\( \circ \) 49\( \frac{1}{1} \) 1.776\( " \).
15. \( \cdot 394536 \) rt. \( \angle \); 35\( \circ \) 30\( \frac{2}{2} \) 29.664\( " \).
16. 2\( \cdot 550809 \) rt. \( \angle \); 229\( \circ \) 34\( \frac{2}{2} \) 22 116\( " \).
17. 7\( \cdot 590005 \) rt. \( \angle \); 683\( \circ \) 6\( \frac{1}{1} \) 1.62\( " \).
28. 5\( \circ \) 33\( \frac{2}{2} \) 20\( " \); 66\( \circ \) 40\( " \).
29. 47\( \frac{7}{7} \)\( \frac{8}{8} \); 42\( \frac{13}{13} \)\( ^\circ \).
31. 33\( \circ \) 20\( " \); 10\( \circ \) 48\( " \).

II. (Page 10.)

1. 25132.74 miles nearly.
2. 19.28 miles per hour nearly.
3. 12.85 miles nearly.
4. 3.14159... inches.
5. 581,194,640 miles nearly.

III. (Pages 13, 14.)

1. 60\( ^\circ \).
2. 240\( ^\circ \).
3. 1800\( ^\circ \).
4. 57\( \circ \) 17\( \frac{44}{44} \)\( ^\circ \).
5. 458\( \circ \) 21\( \frac{58}{58} \)\( ^\circ \).
6. 160\( ^\circ \).
7. \(233^\circ 33' 33.3''\).  
8. \(2000^\circ\).  
9. \(\frac{\pi}{3}\).  
10. \(\frac{221}{360}\pi\).

11. \(\frac{703}{720}\pi\).  
12. \(\frac{3557}{13500}\pi\).  
13. \(\frac{79}{36}\pi\).

14. \(\frac{3\pi}{10}\).  
15. \(1103\).  
16. \(1\cdot726268\pi\).

17. \(81^\circ; 9^\circ\).  
18. \(24^\circ, 60^\circ\), and \(96^\circ\).

19. \(132^\circ 15' 12.6''\).  
20. \(30^\circ, 60^\circ\), and \(90^\circ\).

21. \(\frac{1}{2}, \frac{\pi}{3}\), and \(\frac{2\pi}{3} - \frac{1}{2}\) radians.

22. (1) \(\frac{3\pi}{5}\); 108°.  
(2) \(\frac{5\pi}{7}\); 1284°.

(3) \(\frac{3\pi}{4}\); 135°.  
(4) \(\frac{5\pi}{6}\); 150°.  
(5) \(\frac{15\pi}{17}\); 15814°.

23. 8 and 4.  
24. 10 and 8.  
25. 6 and 8.

26. \(\frac{\pi}{3}\).  
27. (1) \(\frac{5\pi}{12}\) = 75° = 83\ 1/3°;  
(2) \(\frac{7\pi}{18}\) = 70° = 77\ 1/9°;  
(3) \(\frac{5\pi}{8}\) = 112\ 1/2° = 125°.

28. (1) At 7\ \frac{7}{11}\ and 36 minutes past 4; (2) at 28\ \frac{4}{11}\ and 48 minutes past 7.

IV. (Pages 17, 18.)

\[
\left[ \text{Take } \pi = 3.14159 \ldots \text{ and } \frac{1}{\pi} = 31831. \right]
\]

1. \(20.454^\circ\) nearly.  
2. \(\frac{3}{5}\) radian; 34° 22' 38.9''.

3. 68.75 inches nearly.  
4. 0.05236 inch nearly.

5. 24.555 inches nearly.  
6. 1° 25' 57'' nearly.

7. 3959.8 miles nearly.  
8. \(\pi\) ft. = 3.14159 ft.

9. 5 : 4.  
10. 3.1416.

11. \(\frac{4\pi}{35}, \frac{9\pi}{35}, \frac{14\pi}{35}, \frac{19\pi}{35}\), and \(\frac{24\pi}{35}\) radians.

12. 65° 24' 30.4''.  
13. 2062.65 ft. nearly.

14. 1.5359 ft. nearly.  
15. 262.6 ft. nearly.

16. 32142.9 ft. nearly.  
17. 38197.2 ft. nearly.
18. \(19.099\)  
19. 1105.8 miles.
20. 238,833 miles  
21. 21600; 6875.5 nearly.
22. 478 \(\times 10^{11}\) miles.

VI. (Page 31.)

5. \(\sqrt{15} \cdot \frac{1}{4}, \frac{1}{15}, \text{etc.}\)
6. \(\frac{12}{5}, \frac{8}{13}, \frac{7}{60}, \frac{60}{61}, \frac{61}{60}, \frac{8}{5}, \frac{4}{3}\)
7. \(\frac{40}{9}, \frac{41}{40}\)
10. \(\frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{5}{3}\)
11. \(\frac{3}{4}\)
9. \(\frac{15}{17}, \frac{17}{8}\)
13. \(\frac{1}{2}, \sqrt{5}, \frac{3}{5}, \sqrt{5}\)
14. 1 or \(\frac{3}{5}\)
12. \(\frac{3}{5} \text{ or } \frac{5}{13}\)
16. \(\frac{5}{13}\)
17. \(\frac{12}{13}\)
18. \(\frac{1}{\sqrt{3}} \text{ or } 1\)
19. \(\frac{1}{2}\)
20. \(\frac{1}{\sqrt{2}}\)
21. \(1 + \sqrt{2}\)
22. \(\frac{2x(x+1)}{2x^2 + 2x + 1}, \frac{2x + 1}{2x^2 + 2x + 1}\)

VIII. (Pages 44–46.)

1. 34.64... ft.; 20 ft.
2. 160 ft.
3. 225 ft.
4. 136.6 ft.
5. 146.4... ft.
6. 367.9... yards; 454.3... yards.
7. 86.6... ft.
8. 115.359... ft.
9. 87.846... ft.
10. 43.3... ft.; 75 ft. from one of the pillars.
11. 94.641... ft.; 54.641... ft.
12. 1.366... miles.
13. \(30^\circ\)
15. \(13.8564 \text{ miles per hour}\)
16. 25.98... ft.; 70.98... ft.; 85.98... ft.
17. \(32 \sqrt{5} = 71.55... \text{ ft.}\)
19. 10 miles per hour.
20. 86.6... yards.
21. 692.8... yards.

IX. (Page 63.)

1. \(\frac{2250}{6289} \pi, \frac{2500}{6289} \pi \text{ and } \frac{81}{331} \pi \text{ radians.}\)
2. \( 68^\circ 45' 17.8'' \).
4. \( \frac{2xy}{x^3 + y^3}; \frac{2xy}{x^3 - y^3} \).
8. \( \frac{1}{\tan^4 A} - \tan^4 A \).
9. \( \theta = 60^\circ \).
10. In \( 1\frac{1}{2} \) minutes.

X. (Pages 74, 75.)

4. \( -3.66\ldots; 2.3094\ldots \).
5. \( -1.366\ldots; -2.3094\ldots \).
6. 0; 2.
7. 1.4142\ldots; -2.
8. 1.366\ldots; -2.3094\ldots.
9. 45° and 135°.
10. 120° and 240°.
11. 135° and 315°.
12. 150° and 330°.
15. \( -\cos 25^\circ \).
16. \( \sin 6^\circ \).
17. \( -\tan 43^\circ \).
18. \( \sin 12^\circ \).
19. \( \sin 17^\circ \).
20. \( -\cot 24^\circ \).
21. \( \cos 33^\circ \).
22. \( -\cos 28^\circ \).
23. \( \cot 25^\circ \).
24. \( \cos 30^\circ \).
25. \( \cot 26^\circ \).
26. \( -\cosec 23^\circ \).
27. \( -\cosec 36^\circ \).
28. negative.
29. negative.
30. positive.
31. zero.
32. positive.
33. positive.
34. positive.
35. negative.
36. \( \frac{1}{\sqrt{3}} \) and \( -\frac{\sqrt{2}}{\sqrt{3}} \); \( -\frac{1}{\sqrt{3}} \) and \( \frac{\sqrt{2}}{\sqrt{3}} \).

XI. (Pages 83, 84.)

1. \( n\pi + (-1)^n \frac{\pi}{6} \).
2. \( n\pi - (-1)^n \frac{\pi}{3} \).
3. \( n\pi + (-1)^n \frac{\pi}{4} \).
4. \( 2n\pi \pm \frac{2\pi}{3} \).
5. \( 2n\pi \pm \frac{\pi}{6} \).
6. \( 2n\pi \pm \frac{3\pi}{4} \).
7. \( n\pi + \frac{\pi}{3} \).
8. \( n\pi + \frac{3\pi}{4} \).
9. \( n\pi + \frac{\pi}{4} \).
10. \( 2n\pi \pm \frac{\pi}{3} \).
11. \( n\pi + (-1)^n \frac{\pi}{5} \).
12. \( n\pi \pm \frac{\pi}{2} \).
13. \( n\pi \pm \frac{\pi}{3} \).
14. \( n\pi \pm \frac{\pi}{6} \).  
15. \( n\pi \pm \frac{\pi}{3} \).  
16. \( n\pi \pm \frac{\pi}{4} \).

17. \( n\pi \pm \frac{\pi}{6} \).  
18. \( (2n+1)\pi + \frac{\pi}{4} \).  
19. \( 2n\pi - \frac{\pi}{6} \).

20. \(105^\circ \) and \(45^\circ\); \( \left(n + \frac{m}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^m \frac{\pi}{12}, \) and \( \left(\frac{m}{2} - n\right)\pi \mp \frac{\pi}{6} + (-1)^m \frac{\pi}{12}, \)

where \( m \) and \( n \) are any integers.

21. \(187\frac{1}{2}^\circ \) and \(142\frac{1}{2}^\circ\); 
\( \left(n + \frac{m}{2}\right)\pi + \frac{\pi}{8} \pm \frac{\pi}{12} \) and \( \left(n - \frac{m}{2}\right)\pi - \frac{\pi}{8} \mp \frac{\pi}{12}. \)

22. (1) \(60^\circ \) and \(120^\circ\); (2) \(120^\circ \) and \(240^\circ\); (3) \(30^\circ \) and \(210^\circ\).

23. (1) 2; (2) 1; (3) 1; (4) 1; (5) 1.

**XII. (Page 86.)**

1. \( n\pi + (-1)^n \frac{\pi}{6} \).

2. \( 2n\pi \pm \frac{2\pi}{3} \).

3. \( n\pi + (-1)^n \frac{\pi}{3} \).

4. \( \cos \theta = \frac{\sqrt{5} - 1}{2} \).

5. \( n\pi + (-1)^n \frac{\pi}{10} \) or \( n\pi - (-1)^n \frac{3\pi}{10} \) (Art. 120).

6. \( \theta = 2n\pi \pm \frac{\pi}{3} \).

7. \( \theta = n\pi + \frac{\pi}{4} \) or \( n\pi + \frac{\pi}{3} \).

8. \( \theta = n\pi + \frac{2\pi}{3} \) or \( n\pi + \frac{5\pi}{6} \).  

9. \( \tan \theta = \frac{1}{a} \) or \( -\frac{1}{b} \).

10. \( \theta = n\pi \pm \frac{\pi}{4} \).

11. \( \theta = 2n\pi \) or \( 2n\pi + \frac{\pi}{4} \).

12. \( n\pi \pm \frac{\pi}{6} \).

13. \( n\pi \) or \( 2n\pi \pm \frac{\pi}{3} \).

14. \( 2n\pi \pm \frac{\pi}{3} \) or \( 2n\pi \pm \frac{\pi}{6} \).

15. \( \sin \theta = 1 \) or \( -\frac{1}{3} \).
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16. \(\frac{n\pi}{5} + (-1)^n \frac{\pi}{20}\).
17. \(\frac{n\pi}{4}\) or \(\frac{(2n+1)\pi}{10}\).
18. \(2n\pi\) or \(\frac{(2n+1)\pi}{5}\).
19. \(\frac{2r\pi}{m-n}\) or \(\frac{2r\pi}{m+n}\).
20. \((2n+1)\frac{\pi}{5}\) or \(2n\pi - \frac{\pi}{2}\).
21. \(2n\pi\) or \(\frac{2n\pi}{9}\).
22. \((2r + \frac{1}{2})\frac{\pi}{m+n}\) or \((2r - \frac{1}{2})\frac{\pi}{m-n}\).
23. \((n + \frac{1}{2})\frac{\pi}{9}\).
24. \((m + \frac{1}{2})\frac{\pi}{n+1}\).
25. \(\frac{n\pi}{4} \pm \sqrt{1 + \frac{n^2}{16}}\).
26. \(n\pi \pm \frac{\pi}{6}\).
27. \((n + \frac{1}{2})\frac{\pi}{3} \pm \frac{a}{3}\).
28. \((n + \frac{1}{2})\frac{\pi}{4}\).
29. \(\frac{n\pi}{3} \pm \frac{a}{3}\).
30. \(n\pi \pm \frac{\pi}{6}\).
31. \((r + \frac{1}{2})\frac{\pi}{m-n}\).
32. \(\tan \theta = \frac{2n + 1 \pm \sqrt{4n^2 + 4n - 15}}{4}\), where \(n > 1\) or \(-2\).
33. \(\theta = (m + \frac{n}{2})\pi \pm \frac{\pi}{6} + (-1)^n \frac{\pi}{12}\); \(\phi = (m - \frac{n}{2})\pi \pm \frac{\pi}{6} - (-1)^n \frac{\pi}{12}\).
34. \(\frac{1}{5} \left(6m - 4n\right)\pi \pm \frac{\pi}{2} \mp \frac{2\pi}{3}\); \(\frac{1}{5} \left(6n + 4m\right)\pi \pm \frac{\pi}{2} \mp \frac{3\pi}{2}\).
35. 45\(^\circ\) and 60\(^\circ\).
36. \(\frac{1}{3}\) or \(\frac{5}{3}\).
37. \(\pm \frac{1}{3} \sqrt{5}\); \(\pm \frac{1}{2} \sqrt{5}\).

XIII. (Pages 91, 92.)

1. \(-\frac{133}{205}; \frac{84}{205}\).
2. \(-\frac{1596}{3445}; \frac{3444}{3445}\).
3. \(\frac{220}{221}; \frac{171}{221}; \frac{220}{21}\).

XIV. (Pages 96, 97.)

30. \(2 \sin (\theta + n\phi) \sin \frac{3\phi}{2}\).
31. \(2 \sin (\theta + n\phi) \cos \frac{1}{2}\).
XV. (Pages 98, 99.)

1. \( \cos 2\theta - \cos 12\theta. \)
2. \( \sin 12\theta - \sin 2\theta. \)
3. \( \cos 14\theta + \cos 8\theta. \)
4. \( \cos 12^\circ - \cos 120^\circ. \)

XVI. (Page 102.)

1. \( \frac{9}{13}. \)
2. \( \frac{a}{10}. \)

XVII. (Pages 109, 110.)

1. (1) \( \pm \frac{24}{25}; \) (2) \( \pm \frac{120}{169}; \) (3) \( \frac{2016}{4225}. \)
2. (1) \( \frac{161}{289}; \) (2) \( \frac{7}{25}; \) (3) \( \frac{119}{169}. \)
3. \( \alpha. \)

XVIII. (Pages 123—125.)

1. \( \pm \frac{2\sqrt{2} \pm \sqrt{3}}{6}; \) \( \pm \frac{7\sqrt{3} \pm 4\sqrt{2}}{18}. \)
2. \( \pm \frac{13}{12}; \) \( \pm \frac{\sqrt{13}}{2} \) or \( \pm \frac{\sqrt{13}}{3}; \) \( \frac{169}{120}. \)
3. \( \frac{16}{305}; \) \( \frac{49}{305}. \)
4. \( \frac{7}{5\sqrt{2}}. \)
5. \( \pm \frac{1}{3}; \) \( \pm \frac{3}{4}. \)
6. \( \pm \frac{3}{4}. \)
7. \( \frac{\sqrt{4 - \sqrt{2} - \sqrt{6}}}{2\sqrt{2}}; \) \( \frac{\sqrt{4 + \sqrt{2} + \sqrt{6}}}{2\sqrt{2}}; \) \( \sqrt{2} - 1; \)
\( -(\sqrt{2} + 1) + \sqrt{4 + 2\sqrt{2}}. \)
8. \( \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}. \)

23. \( + \) and \( -. \)
24. \( - \) and \( -. \)
25. \( - \) and \( -. \)

29. (1) \( 2n\pi + \frac{\pi}{4} \) and \( 2n\pi + \frac{3\pi}{4}; \)
(2) \( 2n\pi + \frac{3\pi}{4} \) and \( 2n\pi + \frac{5\pi}{4}; \)
(3) \( 2n\pi - \frac{\pi}{4} \) and \( 2n\pi + \frac{\pi}{4}; \)
(4) \( 2n\pi + \frac{\pi}{4} \) and \( 2n\pi + \frac{3\pi}{4}. \)
30. (1) \(2n\pi - \frac{\pi}{4}\) and \(2n\pi + \frac{\pi}{4}\);

(2) \(2n\pi + \frac{3\pi}{4}\) and \(2n\pi + \frac{5\pi}{4}\);

(3) \(2n\pi + \frac{5\pi}{4}\) and \(2n\pi + \frac{7\pi}{4}\).

XIX. (Page 130.)

12. The sine of the angle is equal to \(2\sin 18^\circ\).

13. \(\frac{n\pi}{8}\) or \(\left(2n + \frac{1}{3}\right)\frac{\pi}{8}\).

XXI. (Pages 143, 144.)

1. \(\frac{n\pi}{4}\) or \(\frac{1}{3}\left(2n\pi \pm \frac{\pi}{3}\right)\).

2. \(\left(n + \frac{1}{2}\right)\frac{\pi}{4}\) or \(\left(2n + \frac{1}{3}\right)\frac{\pi}{3}\).

3. \(\left(n + \frac{1}{2}\right)\frac{\pi}{2}\) or \(2n\pi\).

4. \(\left(n + \frac{1}{3}\right)\frac{\pi}{3}\) or \(n\pi + (-1)^n\frac{\pi}{6}\).

5. \(\frac{2n\pi}{3}\) or \(\left(n + \frac{1}{4}\right)\pi\) or \(\left(2n - \frac{1}{2}\right)\pi\).

6. \(\frac{n\pi}{3}\) or \(\left(2n \pm \frac{1}{3}\right)\frac{\pi}{4}\).

7. \(\left(n + \frac{1}{2}\right)\frac{\pi}{2}\) or \(2n\pi \pm \frac{2\pi}{3}\).

8. \(n\frac{\pi}{3}\) or \(\left(n \pm \frac{1}{3}\right)\pi\).

9. \(2n\pi\) or \(\left(\frac{2n}{3} + \frac{1}{2}\right)\pi\).

10. \(n\pi + (-1)^n\frac{\pi}{6}\) or \(n\pi + (-1)^n\frac{\pi}{10}\) or \(n\pi - (-1)^n\frac{3\pi}{10}\).

11. \(\left(n + \frac{1}{2}\right)\frac{\pi}{8}\) or \(\left(n + \frac{1}{2}\right)\frac{\pi}{2}\).

12. \(m\pi\) or \(\frac{1}{n-1}\left[m\pi - (-1)^m\frac{\pi}{6}\right]\).

13. \(2m\pi\) or \(\frac{4m\pi}{n \pm 1}\).

14. \(\frac{2r\pi}{m+n}\) or \(\left(2r + 1\right)\frac{\pi}{m-n}\).

15. \(\left(2r + 1\right)\frac{\pi}{m \pm n}\).

16. \(m\pi\) or \(\frac{m\pi}{n-1}\) or \(\left(m + \frac{1}{2}\right)\frac{\pi}{n}\).

17. \(2n\pi - \frac{\pi}{2}; \frac{1}{5}\left(2n\pi - \frac{\pi}{2}\right)\).
18. \( n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3} \).  
19. \( 2n\pi + \frac{\pi}{4} \).

20. \( n\pi + \frac{\pi}{6} + (-1)^n \frac{\pi}{4} \).  
21. \( 2n\pi + \frac{\pi}{4} \pm A \).

22. \(-21^\circ48' + n \cdot 180^\circ + (-1)^n [68^\circ12'] \).
23. \(2n \cdot 180^\circ + 78^\circ58'; 2n \cdot 180^\circ + 27^\circ18' \).

24. \( n \cdot 180^\circ + 45'; n \cdot 180^\circ + 26^\circ34' \).  
25. \( 2n\pi + \frac{2\pi}{3} \).

26. \( 2n\pi \) or \( 2n\pi + \frac{\pi}{2} \).  
27. \( 2n\pi + \frac{\pi}{2} \) or \( 2n\pi - \frac{\pi}{3} \).

28. \( 2n\pi + \frac{\pi}{6} \).  
29. \( n\pi \).

30. \( \sin \theta = \pm \sqrt{17} - 1 \).  
31. \( \cos \theta = \frac{\sqrt{17} - 3}{4} \).

32. \( n\pi + \frac{\pi}{3} \) or \( n\pi + \frac{\pi}{2} \).  
33. \( 2n\pi + \frac{\pi}{3}; 2n\pi + \frac{\pi}{4} \).

34. \( \left( n + \frac{1}{4} \right) \frac{\pi}{2} \).  
35. \( n\pi \pm \frac{\pi}{4} \).  
36. \( n\pi + \frac{\pi}{4} \).

37. \( \theta = \frac{n\pi}{2} \) or \( n\pi \pm \frac{\pi}{3} \); also \( \theta = n\pi \pm \frac{a}{2} \), where \( \cos a = \frac{1}{3} \).

38. \( \left( n + \frac{1}{3} \right) \frac{\pi}{3} \).  
39. \( n\pi \pm \frac{\pi}{3} \).

XXIII. (Pages 157, 158.)

1. 1.90309; 3.4771213; 2.0334239; 1.4650389.
2. 1.553361; 2.1241781; 5.388340; 1.0759623.
3. 2; 2; 0; 4; 2; 0; 3. 4. 312936.
5. 1.32057; 5.88453; 461791.
6. (1) 21; (2) 13; (3) 30; (4) the 7th; (5) the 21st; (6) the 32nd.
7. (1) \( \frac{4b}{c-b-a} \); (2) \( \frac{a+2b}{4c-3b-2a} \); (3) \( \frac{4a+7b}{a+3b-2c} \);
(4) \( \frac{2b(2a-b)}{5ab+3ac-2b^2-bc} \) and \( \frac{2ab}{5ab+3ac-2b^2-bc} \),
where \( a = \log 2, b = \log 3, \) and \( c = \log 7 \).
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8. \(0.22221\). 9. \(8.6415\). 10. \(9.6192\).
11. \(1.6389\). 12. \(4.7162\). 13. \(0.41431\).

XXIV. (Pages 168—170.)

1. \(4.5527375; 1.5527394\).
2. \(4.7689529; 3.7689502\).
3. \(478.475; 0.04784777\). 4. \(2.583674; 0.0258362\).
5. (1) \(4.7204815\); (2) \(2.7220462\); (3) \(4.7240079\);
   (4) \(5.273.63\); (5) \(0.05296726\); (6) \(5.26064\).
6. \(0.6870417\). 7. \(43^\circ 23'45''\).
8. \(0.8455104; 0.8454509\). 9. \(32^\circ 16'35''; 32^\circ 16'21''\).
10. \(4.1203060; 4.1218748\).
11. \(4.3993263; 4.3976823\). 12. \(13^\circ 8'47''\).
13. \(9.9147334\). 14. \(34^\circ 44'27''\).
15. \(9.5254497; 71^\circ 27'43''\). 16. \(10.0229414\).
17. \(18^\circ 27'17''\). 18. \(36^\circ 52'12''\).

XXV. (Pages 172, 173)

1. \(13^\circ 27'31''\). 2. \(22^\circ 1'28''\).
3. \(1.0997340; 65^\circ 24'12.5''\).
4. \(9.6198509; 22^\circ 36'28''\).
5. \(10^\circ 15'34''\). 6. \(44^\circ 55'55''\).
7. (1) \(9.7279043\); (2) \(9.9270857\); (3) \(10.1958917\);
   (4) \(10.0757907\); (5) \(10.2001337\);
   (6) \(10.0725027\); (7) \(9.7245162\).
8. (1) \(57^\circ 30'24''\); (2) \(57^\circ 31'58''\); (3) \(32^\circ 31'15''\);
   (4) \(57^\circ 6'39''\).
9. \(0.5373602\).
10. (1) \(\cos (x - y) \sec x \sec y\); (2) \(\cos (x + y) \sec x \sec y\);
    (3) \(\cos (x - y) \cosec x \sec y\);
    (4) \(\cos (x + y) \cosec x \sec y\);
    (5) \(\tan^2 x\); (6) \(\tan x \tan y\).
ANSWERS.

XXVI. (Pages 180, 181.)

1. \( \frac{1}{5}, \frac{1}{2}, \text{ and } \frac{9}{7} \).

2. \( \frac{4}{\sqrt{41}}, \frac{3}{5}, \text{ and } \frac{8}{\sqrt{41}}; \frac{40}{25}, \text{ and } \frac{496}{1025} \).

3. \( \frac{3}{5}, \frac{4}{5}, \text{ and } 1 \).

4. \( \frac{5}{12}, \frac{12}{5}, \text{ and } \infty \).

5. \( \frac{4}{5}, \frac{56}{65}, \text{ and } \frac{12}{13} \).

6. \( \frac{7}{41} \text{ and } \frac{287}{816} \).

7. \( 60^\circ, 45^\circ, \text{ and } 75^\circ \).

XXVII. (Pages 186—188.)

23. \( 16\frac{1}{2} \text{ ft.} \)

25. \( \frac{2}{5} \).

28. \( \frac{313}{338} \).

XXVIII. (Page 191.)

1. \( 186.60... \text{ and } 193.18 \).

2. \( 26^\circ 33'54''; 63^\circ 26'6''; 10\sqrt{5} \text{ ft.} \).

3. \( 48^\circ 35'25''; 36^\circ 52'12'' \text{ and } 94^\circ 32'23'' \).

4. \( 75^\circ \text{ and } 15^\circ \).

XXIX. (Pages 194, 195.)

1. \( 90^\circ \).

2. \( 30^\circ \).

4. \( 120^\circ \).

5. \( 45^\circ, 120^\circ \text{ and } 15^\circ \).

6. \( 45^\circ, 60^\circ, \text{ and } 75^\circ \).

7. \( 58^\circ 59'33'' \).

8. \( 77^\circ 19'11'' \).

9. \( 76^\circ 39'5'' \).

10. \( 104^\circ 28'39'' \).

11. \( 56^\circ 15'4'', 59^\circ 51'10'' \text{ and } 63^\circ 53'46'' \).

12. \( 38^\circ 56'33'', 47^\circ 41'7'' \text{ and } 93^\circ 22'20'' \).

13. \( 130^\circ 42'20.5'', 23^\circ 27'8.5'', \text{ and } 25^\circ 50'31'' \).

XXX. (Pages 199—201.)

1. \( 63^\circ 13'2''; 43^\circ 58'28''. \)

2. \( 117^\circ 38'45''; 27^\circ 38'45'' \).

3. \( 8\sqrt{7} \text{ feet}; 79^\circ 6'24''; 60^\circ; 40^\circ 53'36'' \).

4. \( 87^\circ 27'25.5''; 32^\circ 32'34.5''. \)

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5.  $40^\circ 53'36''; 19^\circ 6'24''; \sqrt{7} : 2.$
6.  $71^\circ 44'30''; 48^\circ 15'30''.$  7.  $78^\circ 17'40''; 49^\circ 36'20''.$
8.  $108^\circ 12'26''; 49^\circ 27'34''.$
9.  $A = 45^\circ; B = 75^\circ; c = \sqrt{6}.$  10.  $\sqrt{6}; 15^\circ; 105^\circ.$
15.  $7.589467; 108^\circ 26'6''; 18^\circ 26'6''; 53^\circ 7'48''.$
16.  2.529823.  17.  226.87; 73° 34'50''; 39° 45'10''.
18.  $A = 83^\circ 7'30''; B = 42^\circ 16'21''; c = 199.099.$
19.  $B = 110^\circ 48'15''; C = 26^\circ 56'15''; a = 93.5192.$
20.  $73°1'51''$ and $48°41'9''.$
21.  $88°30'1''$ and $33°30'59''.$

XXXI. (Pages 207—209.)

1. There is no triangle.
2. $B_1 = 30^\circ, C_1 = 105^\circ,$ and $b_1 = \sqrt{2}; B_2 = 60^\circ, C_2 = 75^\circ,$ and $b_2 = \sqrt{6}.$
3. $B_1 = 15^\circ, C_1 = 135^\circ$ and $b_1 = 50 (\sqrt{6} - \sqrt{2}); B_2 = 105^\circ,$ $C_2 = 45^\circ,$ and $b_2 = 50 (\sqrt{6} + \sqrt{2}).$
5. $4\sqrt{3} \pm 2\sqrt{5}.$
6.  100$\sqrt{3};$ the triangle is right-angled.
8.  $33° 29'30''$ and $101° 30'30''.$
9.  17.1 or 3.68.
10. (1) The triangle is right-angled and $B = 60^\circ.$
    (2) $b_1 = 60.3893, B_1 = 8^\circ 41' and C_1 = 141^\circ 19';$
    $B_2 = 111^\circ 19'$ and $C_2 = 38^\circ 41'.'$
11.  $65^\circ 59'$ and $41^\circ 56'12''.$
12.  5.988... and 2.6718... miles per hour.
13.  $63^\circ 2'12''$ or $116^\circ 57'48''.$
14.  $62^\circ 31'23''$ and $102^\circ 17'37'', or 117^\circ 28'37''$ and $47^\circ 20'23''.$
15.  5926.61.

XXXII. (Page 210.)

1.  $7:9:11.$  4.  79.063.
5.  1 mile; 1.219714... miles.  7.  20.97616... ft.
8.  6.85673... and 5.4378468... feet.  9.  404.4352 ft.
10.  233.2883 yards.  11.  2229 yards.
XXXIII. (Pages 215—218.)

1. 100 ft. high and 50 ft. broad; 25 feet.
2. 25.7834 yds. 3. 33.07... ft.; 17$\frac{1}{2}$ ft.
4. 18.3... ft. 5. 120 ft. 6. $h \tan \alpha \cot \beta$.
7. 1939.2... ft. 8. 100 ft. 9. 61.224... ft.
10. 100$\sqrt{2}$ ft.
15. $PQ = BP = BQ = 1000$ ft.; $AP = 500 (\sqrt{6} - \sqrt{2})$ ft;
    $AQ = 1000 \sqrt{2}$ ft.
16. .32119 miles. 17. .1736482 miles; .9848078 miles.
18. 119.2862 ft. 19. 132.266 ft.
22. 125.3167 ft.

XXXIV. (Pages 222—227.)

3. 20 ft.; 40 ft.
4. $l \cosec \gamma$, where $\gamma$ is the sun's altitude; $\sin \gamma = \frac{2}{7}$.
5. 3.732... miles; 12.342... miles per hour at an angle, whose tangent is $\sqrt{3} + 1$, S. of E.
6. 10.2426... miles per hour.
7. 16.3923... miles; 14.697... miles.
8. 2.39 miles; 1.366 miles.
9. It makes an angle whose tangent is $\frac{2}{3}$; $\frac{9}{52}$ hour.
13. $c \sin \beta \cosec (\alpha + \beta)$; $c \sin \alpha \sin \beta \cosec (\alpha + \beta)$.
14. 9 yds.; 2 yds.
16. $\frac{a}{3}$; $\frac{2a}{3}$.
20. At a distance $\frac{375}{\sqrt{7}}$ ft. from the cliff.
21. $c(1 - \sin \alpha) \sec \alpha$. 22. 114.4123 ft. 24. 1069.745645 ft.
26. The angle whose tangent is $\frac{1}{2}$.
29. 45°.
32. 18°26'6". 34. $\tan \alpha \sec \beta : 1$.
37. 91.896 ft. 38. 1960.95 yds.
39. 2.45832 miles. 40. 333.4932 ft.

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XXXV. (Pages 229, 230.)

1. 84.  2. 216.  3. 630.  4. 3720.
5. 270.  6. 117096.  7. 1470.
8. 1.183....  12. 35 yds. and 26 yds.
13. 14.941... inch.  14. 5, 7, and 8 ft.  15. 120°.
17. 45° and 105°; 135° and 15°.
18. 17.1064... sq. ins.

XXXVI. (Pages 237, 238.)

3. 8\frac{1}{6}, 1\frac{1}{2}, 8, 2, and 24 respectively.

XXXVII. (Pages 247—250.)

35. 2.1547... or \cdot1547 times the radius of each circle.

39. \[ A_n = \frac{\pi}{3} + (-1)^n \cdot 2^n \left( A - \frac{\pi}{3} \right), \ldots. \]

XXXVIII. (Pages 255—257.)

1. (1) \(3\sqrt{105}\) sq. ft.; (2) \(10\sqrt{7}\) sq. ft.  3. 1\frac{5}{7} and 2\frac{1}{2} ft.

XXXIX. (Pages 259—261.)

1. 77.98 ins.  2. 5359.
3. (1) 1.720... sq. ft.; (2) 2.598... sq. ft.;
(3) 4.8284... sq. ft.; (4) 7.694... sq. ft.;
(5) 11.196... sq. ft.
4. 1.8866... sq. ft.  5. 3.3136... sq. ft.
6. 2 + \sqrt{2} : 4; \sqrt{2} + \sqrt{2} : 2.  12. 3.
14. 6.  15. 9.  16. 20 and 10.
17. 6 and 5, 12 and 8, 18 and 10, 22 and 11, 27 and 12, 42 and 14, 54 and 15, 72 and 16, 102 and 17, 162 and 18, 342 and 19 sides respectively.

19. \frac{2}{3}\sqrt{3}; \sqrt{6}.

XL. (Pages 266, 267.)

1. -.00204.  2. -.00007.  3. -.00029.
4. -.99999.  5. 25783.10077.  6. 1.0000011.
7. 34° 23".  8. 28° 40' 37".  9. 39° 42'.
10. 2° 33' 44".  11. 114.59... inches.
XLI. (Pages 269, 270.)
1. 435.77 sq. ft. 2. 4.9087... sq. ft.
3. 127° 19’ 26”.
4. 6 sq. ft.
5. 11.0004 inches.
6. 0.00044625 inch.
7. \( \frac{2}{3} \pi r \).

XLII. (Pages 271, 272.)
1. 1° 8’ 45”.
2. 17.23 miles.
3. 61 miles; 1° 48’ nearly.
4. About 61800 metres = about 38 \( \frac{1}{2} \) miles.
5. 3960 miles.

XLIII. (Pages 279—281.)
28. \( \pm \sqrt{\sin 2\beta} \).
29. \( \frac{1}{6} \).
30. \( \pm \frac{1}{\sqrt{2}} \).
31. \( 4 \sqrt{\frac{3}{7}} \).
32. \( \frac{1}{4} \).
33. \( n \pi \), or \( n \pi + \frac{\pi}{4} \).
34. \( \sqrt{3} \).
35. \( \sqrt[5]{3} \).
36. \( \sqrt{3} \) or \(- (2 + \sqrt{3})\).
37. \( \sqrt{3} \) or \( 2 - \sqrt{3} \).
38. \( n \), or \( n^2 - n + 1 \).
39. \( \frac{1}{2} \sqrt{\frac{3}{7}} \).
40. 13.
41. \( x \) is given by the equation \( x^4 - x^2 (ab + ac + ad + bc + bd + cd) + abcd = 0 \).
42. \( x = ab \).
43. \( ab + [\sqrt{a^2 - 1} + \sqrt{b^2 - 1}] \).
44. \( \frac{a - b}{1 + ab} \).

XLIV. (Pages 287—289.)
1. \( \frac{1}{2} \sin 2n\theta \csc \theta \).
2. \( \cos \frac{3n - 1}{4} A \sin \frac{3n}{4} A \csc \frac{3}{4} A \).
6. \( \frac{1}{2} \).
7. \( \sin \left[ a + \left( n - \frac{1}{2} \right) \beta \right] \sin n\beta \sec \frac{\beta}{2} \).
8. \( - \sin \frac{n\theta}{n - 2} \).
9. \( \sin 2nx (\cos 2nx + \sin 2nx) (\cos x + \sin x) \csc 2x \).
10. \( \frac{1}{4} \left[ (n + 1) \sin 2a - \sin (2n + 2) a \right] \csc a \).
11. \( \frac{1}{2} \sin (2n + 2) \alpha \cdot \sin 2na \csc \alpha. \)

12. \( \frac{n}{2} \cos 2a - \frac{1}{2} \cos (n + 3) \alpha \sin na \csc \alpha. \)

13. \( \frac{\cos (2na - a) \cos (n+1) \beta - \cos (2na + a) \cos \alpha \cos \beta + \cos a (1- \cos \beta)}{2 (\cos \beta - \cos 2a)} \)

14. \( \frac{1}{4} [(2n + 1) \sin \alpha - \sin (2n + 1) \alpha] \csc \alpha. \)

15. \( \frac{n}{2} - \frac{1}{2} \cos [2 \theta + (n - 1) \alpha] \sin na \csc \alpha. \)

16. \( \frac{3}{4} \sin \frac{n+1}{2} a \sin \frac{na}{2} \csc \alpha - \frac{1}{4} \sin \frac{n+1}{2} a \cdot \sin \frac{3na}{2} \csc \frac{3a}{2}. \)

17. \( \frac{1}{8} [3n - 4 \cos (n+1) \alpha \sin na \csc \alpha + \cos (2n+2) \alpha \sin 2na \csc 2\alpha]. \)

18. \( \frac{1}{8} [3n + 4 \cos (n+1) \alpha \sin na \csc \alpha + \cos (2n+2) \alpha \sin 2na \csc 2\alpha]. \)

19. \( \frac{1}{4} \sin \frac{n \theta}{2} \left[ \cos \frac{n-1}{2} \theta + \cos \frac{n+3}{2} \theta + \cos \frac{n+7}{2} \theta \right] \csc \frac{\theta}{2} + \frac{1}{4} \sin \frac{3n \theta}{2} \cos \frac{3n+9}{2} \theta \csc \frac{3 \theta}{2}. \)

20. \( -\frac{1}{2} \sin (2a + 2n \beta) \sin 2n \beta \sec \beta. \)

XLV. (Pages 293, 294.)

1. \( a^2 + b^2 = c^2 + d^2. \)

2. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos (a - \beta) = \sin^2 (a - \beta). \)

3. \( a (2c^2 - d^2) = b dc. \)

4. \( a \sin \alpha + b \cos \alpha = \sqrt{2b (a + b)}. \)

5. \( \frac{x^2}{a} + \frac{y^2}{b} = 1. \)

6. \( \frac{x}{a} + \frac{y}{b} = \alpha + b. \)

7. \( (p^2 + 1)^2 + 2q (p^2 + 1) (p + q) = 4 (p + q)^2. \)

8. \( (a^2 + y^2 - b^2)^2 = a^2 [(x + b)^2 + y^2]. \)

9. \( a^2 + b^2 = 2 + 2 \cos \alpha. \)

10. \( \alpha \beta = a \{4b^2 + (b^2 - c^2)^2\}. \)

11. \( x (c^2 - a^2 - b^2) = y \sqrt{(a+b+c)(-a+l+c)(a-b+c)(a+b-c)}. \)

12. \( xy = (y-x) \tan \alpha. \)

13. \( a^2 (a - c) (a - d) = b^2 (b - c) (b - d). \)

14. \( 8bc = a \{4b^2 + (b^2 - c^2)^2\}. \)

15. \( b^2 [x (b^2 - a^2) + a (a^2 + b^2)] = 4c^4 [b^2 a^2 + a^2 y^2]. \)
MISCELLANEOUS EXAMPLES. (Pages 301—326.)

4. 142 ft. approx.; 4° 30’ approx.
5. 41, 50, and 21 feet.
7. \( \sin (\beta - \alpha) = \pm \sqrt{1 - b} \sqrt{1 - a^2} \mp a \sqrt{b} \).
9. \( n\pi + 38^\circ 7' 27'' \).
10. \( \left(n + \frac{1}{4}\right) \frac{\pi}{3} \).
14. 7 - 3\sqrt{5} : 2.
21. (i) \( \theta = n\pi + (-1)^n \frac{a + \beta}{2} \),
or \( \tan \theta = \left(1 - \cosec a \cosec \beta\right) \tan \frac{a + \beta}{2} \);
(ii) \( \theta = n\pi \) or \( \left(n + \frac{1}{3}\right) \frac{\pi}{3} \).
22. 51° 19’; 78° 28’; 108° 13’.
23. 1298 feet nearly; 13° 31’ East of South.
28. 80 feet.
30. \( \frac{1}{2} \tan^{-1} x; \frac{x + y}{1 - xy} \).
31. \( (l^2 + m^2)(1 - n) = 2m(1 + n) \).
32. Two roots.
35. \( \left(m \pm \frac{1}{12}\right) \pi; \left(n \pm \frac{1}{6}\right) \pi \).
47. \( (\lambda^2 - 1)^2 = 27\lambda^2 \cos^2 a \sin^2 a \).
48. 1.39 radians = 79° 30’ nearly.
49. \( \sin^2 (\beta - \gamma) \sec^2 (a - \beta) \sec^2 (a - \gamma) \).
52. 11° 12’ North of East.
54. Six values.
58. \( \frac{1}{3} \left[ n\pi + \frac{\pi}{2} - a - \beta - \gamma \right] \).
59. 72.77 feet.
62. \( c\sqrt{2a - b} = a\sqrt{2a} - (a - b) \sqrt{b} \).
63. 118\frac{1}{2}°.
64. -1° 19’, + 28\frac{1}{2}’, and + 50\frac{1}{2}’ nearly.
69. \( \frac{a \sin a \sin \beta}{\sqrt{\sin (\beta - a) \sin (\beta + a)}} \).
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73. \[ \cos (a + \beta + \gamma + \delta) + \cos (a + \beta - \gamma - \delta) \\
+ \cos (a - \beta + \gamma - \delta) + \cos (a - \beta - \gamma + \delta) - \cos (-a + \beta + \gamma + \delta) \\
- \cos (a - \beta + \gamma + \delta) - \cos (a + \beta - \gamma + \delta) - \cos (a + \beta + \gamma - \delta). \]

76. \[ 66^\circ 19^{1}_2 \text{ approx.} \]

80. \[ x = -2.4531. \]

82. \[ \tan \theta = 2 \pm \sqrt{11} \text{ or } 2 \pm \sqrt{3}; \]
\[ \tan \phi = 2 \pm \sqrt{11} \text{ or } 2 \pm \sqrt{3}. \]

84. \[ 16.47 \text{ miles.} \]

87. \[ 27y^2 = x^2(9 - 8x^2). \]

94. \[ 2a^3 + c = 3a(1 + b). \]

95. \[ \sin \frac{n\theta}{2} \sin \left(\frac{n + 3}{2}\right) \theta \left[1 + 2 \cos 2\theta\right] \cos e \frac{\theta}{2} \\
- \sin \frac{3n + 9}{2} \theta \sin \frac{3n\theta}{2} \cos e \frac{3\theta}{2}; \]

96. \[ 2.55 \text{ radians } = 146^\circ 6' \text{ nearly.} \]

104. \[ \tan \theta = 0, 1, -1, -2. \]

108. \[ 2379 \text{ feet.} \]

117. \[ 11^\circ 27'. \]

120. \[ x^2 + y^2 = a^2 + b^2. \]

128. \[ (a^2 + b^2)^2 - 4a^2 = \frac{8ab}{c}. \]

136. \[ (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2. \]

138. \[ mc_1 + na_1 + \sqrt{\frac{m^2c_2^2 + n^2a_2^2 + 2mn a_2 c_2 \cos (a - \gamma) \sin (nt + \beta)}}{m + n}, \]

where \( \tan \beta = \left(\frac{mc_2 \sin \gamma + na_2 \sin a}{mc_2 \cos \gamma + na_2 \cos a}\right). \]

146. \[ m^2 + m \cos a = 2. \]

148. \[ 77\frac{1}{3}^\circ; 63\frac{1}{2}^\circ. \]

151. \[ (x^2 + y^2 + 2ax)^2 = 4a^2(x^2 + y^2). \]

161. \[ (1 - x^2 - y^2) \sin (a - \beta) \sin (\beta - \gamma). \]

176. \[ a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 = 0. \]

177. \[ a^2y^2 - 4xy = (x + y)^2 \tan^2 (\beta - a). \]

178. \[ b^2 (\sqrt{3}x - y)^2 \{6(x^2 - y^2) - b^2\} = 8(x^2 - y^2)^3 + 4b^2(x^2 - y^2). \]

181. \[ 3 \text{ roots}; 299^9 \text{ approx.} \]

182. \[ .98 \text{ radian } = 56^\circ 9' \text{ nearly.} \]

185. \[ 2.07 \text{ and } -1.23. \]
ERRATUM

In the last line of the Tables on Pages xxv, xxvii, xxix, xxxi, xxxiii, xxxv, xxxvii, xxxix

for 50' 40' 30' 20' 10' 0'
read 60' 50' 40' 30' 20' 10'

TABLES OF LOGARITHMS, NATURAL SINES, NATURAL TANGENTS, LOGARITHMIC SINES, AND LOGARITHMIC TANGENTS.
### TABLE I.

LOGARITHMS OF NUMBERS.

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| 11 | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 |

| 11 | 21 | 32 | 43 | 54 | 64 | 75 | 86 | 97 |
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| 10 | 20 | 31 | 41 | 51 | 61 | 71 | 82 | 92 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
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| 10 | 19 | 29 | 38 | 48 | 57 | 67 | 76 | 86 |
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| 9  | 17 | 25 | 34 | 43 | 52 | 60 | 69 | 77 |
| 8  | 17 | 25 | 34 | 42 | 51 | 59 | 67 | 76 |
| 8  | 17 | 25 | 33 | 42 | 50 | 58 | 66 | 75 |
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| 8  | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
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**Table I. Logarithms of Numbers.**

0 1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7 8 9
### NATURAL SINES.

|       | 0°  | 1°  | 2°  | 3°  | 4°  | 5°  | 6°  | 7°  | 8°  | 9°  | 10° | 11° | 12° | 13° | 14° | 15° | 16° | 17° | 18° | 19° |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0°    | 0.00000 | 0.00291 | 0.00582 | 0.00873 | 0.01164 | 0.01454 |
| 1°    | 0.01745 | 0.02036 | 0.02327 | 0.02618 | 0.02908 | 0.03199 |
| 2°    | 0.03490 | 0.03781 | 0.04071 | 0.04362 | 0.04653 | 0.04943 |
| 3°    | 0.05234 | 0.05524 | 0.05814 | 0.06105 | 0.06395 | 0.06685 |
| 4°    | 0.06976 | 0.07266 | 0.07556 | 0.07846 | 0.08136 | 0.08426 |
| 5°    | 0.08716 | 0.09005 | 0.09295 | 0.09585 | 0.09874 | 0.10164 |
| 6°    | 0.10453 | 0.10742 | 0.11031 | 0.11320 | 0.11609 | 0.11898 |
| 7°    | 0.12187 | 0.12476 | 0.12764 | 0.13053 | 0.13341 | 0.13629 |
| 8°    | 0.13917 | 0.14205 | 0.14493 | 0.14781 | 0.15069 | 0.15356 |
| 9°    | 0.15643 | 0.15931 | 0.16218 | 0.16505 | 0.16792 | 0.17078 |
| 10°   | 0.17365 | 0.17651 | 0.17937 | 0.18224 | 0.18509 | 0.18795 |
| 11°   | 0.19081 | 0.19366 | 0.19652 | 0.19937 | 0.20222 | 0.20507 |
| 12°   | 0.20791 | 0.21076 | 0.21360 | 0.21644 | 0.21928 | 0.22212 |
| 13°   | 0.22495 | 0.22778 | 0.23062 | 0.23345 | 0.23627 | 0.23910 |
| 14°   | 0.24192 | 0.24474 | 0.24756 | 0.25038 | 0.25320 | 0.25601 |
| 15°   | 0.25882 | 0.26163 | 0.26443 | 0.26724 | 0.27004 | 0.27284 |
| 16°   | 0.27564 | 0.27843 | 0.28123 | 0.28402 | 0.28680 | 0.28959 |
| 17°   | 0.29237 | 0.29515 | 0.29793 | 0.30071 | 0.30348 | 0.30625 |
| 18°   | 0.30902 | 0.31178 | 0.31454 | 0.31730 | 0.32006 | 0.32282 |
| 19°   | 0.32557 | 0.32832 | 0.33106 | 0.33381 | 0.33655 | 0.33929 |

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### NATURAL TANGENTS.

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**Notes:**
- The table continues with similar entries for angles up to 90°.
- The values are typically used in trigonometry and calculus for calculating slopes and other geometric properties.

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23-2
## Natural Tangents

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The cotangent of a small angle of \( n' \) or the tangent of \( 90° - n' \) is very nearly equal to 3437.7 divided by \( n \).
### LOGARITHMIC SINES.

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Differences vary so rapidly here that tabulation is impossible. For small angles of $n$ minutes log sine $n'$ or log cosine $(90° - n') = \log n + 4.46373$. 

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**Table IV. LOGARITHMIC SINES.**
### LOGARITHMIC SINES.

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**Note:** The table continues in a similar format for other angles and logarithmic values.
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Differences vary so rapidly here that tabulation is impossible.

For small angles of $n$ minutes $\log \tan n' = \log n + 4.46373$.

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Table V: Logarithmic Tangents.
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This table provides the logarithmic tangents for various angles. The values are approximate and are used in calculations involving logarithms and trigonometry. The differences shown in the table indicate how the values change as the angle increases, but tabulation is impossible to show here due to the nature of logarithmic functions.
CONSTANTS.

One Radian = 57° 14' 45" nearly = 206265"
log 206265 = 5.3144255.

$\pi = 3.14159265.$
$\log \pi = 0.4971499.$

$\frac{1}{\pi} = 0.31830989.$
$\log \frac{1}{\pi} = 1.5028501.$

$\frac{\pi}{180} = 0.01745329.$
$\log \frac{\pi}{180} = 2.2418774.$

$\frac{180}{\pi} = 57.2957795.$
$\log \frac{180}{\pi} = 1.7581226.$

$\pi^2 = 9.86960440.$
$\log \pi^2 = 0.9942997.$

$\frac{1}{\pi^2} = 0.10132118.$
$\log \frac{1}{\pi^2} = 1.0057003.$

$\sqrt{\pi} = 1.77245385.$
$\log \sqrt{\pi} = 0.2485749.$

$\frac{1}{\sqrt{\pi}} = 0.56418958.$
$\log \frac{1}{\sqrt{\pi}} = 1.7514251.$

$\frac{1}{\sqrt[3]{\pi}} = 1.46459189.$
$\log \frac{1}{\sqrt[3]{\pi}} = 0.1657166.$

$\frac{1}{\sqrt[3]{\pi}} = 0.68278406.$
$\log \frac{1}{\sqrt[3]{\pi}} = 1.8342834.$

$\sqrt{2} = 1.4142135 \ldots$
$\sqrt{3} = 1.7320508 \ldots$

$\sqrt{5} = 2.2360679 \ldots$
$\sqrt{6} = 2.4494989 \ldots$

$\sqrt{7} = 2.6457513 \ldots$
$\sqrt{8} = 2.8284271 \ldots$

$\sqrt{10} = 3.1622776 \ldots$
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